2.2 Limits Involving Infinity

Horizontal Asymptote

$y = b$ is a horizontal asymptote if either $\lim\limits_{x \to \infty} f(x) = b$ or $\lim\limits_{x \to -\infty} f(x) = b$

Vertical Asymptote

$x = a$ is a vertical asymptote if either $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or $\lim\limits_{x \to a^-} f(x) = \pm\infty$

End Behavior Model

g is

(a) a right end behavior model for f if and only if $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$

(b) a left end behavior model for f if and only if $\lim\limits_{x \to -\infty} \dfrac{f(x)}{g(x)} = 1$

Inequalities Relating Some Important Functions

As $x \to \infty$, $\log_a x < x^b < c^x$ for all constants $a > 1$, $b > 0$, $c > 1$

For problems 1 – 4, find (a) $\lim\limits_{x \to \infty} f(x)$ (b) $\lim\limits_{x \to -\infty} f(x)$, and

(c) identify all horizontal asymptotes

1. $f(x) = \dfrac{\sqrt{|x|}}{2^{-x}}$

(a) $\lim\limits_{x \to \infty} \dfrac{\sqrt{|x|}}{2^{-x}} \to \dfrac{\infty}{0^+} \to \boxed{\infty}$

(b) $\lim\limits_{x \to -\infty} \dfrac{\sqrt{|x|}}{2^{-x}} \to \dfrac{\infty}{\infty} \to \boxed{0}$ (c) $\boxed{y = 0}$

2. $f(x) = \dfrac{4x - 3}{|x - 5|}$

(a) $\lim\limits_{x \to \infty} \dfrac{4x - 3}{|x - 5|} = \boxed{4}$ (b) $\lim\limits_{x \to -\infty} \dfrac{4x - 3}{|x - 5|} = \boxed{-4}$

(c) $\boxed{y = 4, \quad y = -4}$

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3. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

(a) $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{x^2 + 1}} = \boxed{1}$

(b) $\lim\limits_{x \to -\infty} \dfrac{x}{\sqrt{x^2 + 1}} = \boxed{-1}$

(c) $\boxed{y = 1, \quad y = -1}$

4. $f(x) = \dfrac{3x + \sin(3x)}{x}$

(a) $\lim\limits_{x \to \infty} \dfrac{3x + \sin(3x)}{x} = \lim\limits_{x \to \infty} \dfrac{3x}{x} + \lim\limits_{x \to \infty} \dfrac{\sin(3x)}{x}$

$= 3 + \dfrac{[-1, 1]}{\infty} = \boxed{3}$

(b) $\lim\limits_{x \to -\infty} \dfrac{3x + \sin(3x)}{x} = \lim\limits_{x \to -\infty} \left(\dfrac{3x}{x} + \dfrac{\sin(3x)}{x} \right) = \boxed{3}$

(c) $\boxed{y = 3}$

For problems 5 – 8, find the limits.

5. $\lim\limits_{x \to 0^+} \dfrac{\llbracket x \rrbracket}{x}$

$\to \dfrac{0}{0^+} \to \boxed{0}$

6. $\lim\limits_{x \to 0^+} \dfrac{x}{\llbracket x \rrbracket}$

$\to \dfrac{0^+}{0} \to \boxed{\text{Does Not Exist}}$

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7. $\lim\limits_{x \to 0^-} \cot x$

$= \lim\limits_{x \to 0^-} \dfrac{\cos x}{\sin x} \to \dfrac{1}{0^-} \to \boxed{-\infty}$

8. $\lim\limits_{x \to -3^-} \dfrac{-2}{x + 3}$

$\to \dfrac{-2}{0^-} \to \boxed{\infty}$

For problems 9 – 11, find the vertical asymptotes of the graph of f.

9. $f(x) = \dfrac{3}{x^3 - 8}$

$$= \frac{3}{(x-2)(x^2+2x+4)}$$

$$\to \lim_{x \to 2^-} \frac{3}{(x-2)(x^2+2x+4)}$$

$$\to \frac{3}{12(0^-)} \to -\infty \qquad \text{and}$$

$$\lim_{x \to 2^+} \frac{3}{(x-2)(x^2+2x+4)}$$

$$\to \frac{3}{12(0^+)} \to \infty \qquad \text{so}$$

$$\boxed{\text{vertical asymptote at } x = 2}$$

10. $f(x) = \csc x$

$$= \frac{1}{\sin x} \to \lim_{x \to 0^-} \frac{1}{\sin x}$$

$$\to \frac{1}{0^-} \to -\infty \qquad \text{and}$$

$$\lim_{x \to 0^+} \frac{1}{\sin x} \to \frac{1}{0^+} \to \infty$$

$$\boxed{\begin{array}{l}\text{so vertical asymptotes at} \\ y = k\pi, \text{ where k is any integer}\end{array}}$$

11. $f(x) = \dfrac{3 + x}{2x^2 + 5x - 3}$

$$= \frac{3+x}{(2x-1)(x+3)} \to \frac{1}{2x-1} \qquad \text{so}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^-} \frac{1}{2x-1} \to \frac{1}{0^-} \to -\infty$$

$$\lim_{x \to \left(\frac{1}{2}\right)^+} \frac{1}{2x-1} \to \frac{1}{0^+} \to \infty$$

$$\boxed{\text{vertical asymptote at } x = \frac{1}{2}}$$

For problems 12 – 15, find (a) a simple basic function as a right end behavior model, and (b) a simple basic function as a left end behavior model

12. $f(x) = \dfrac{3x^4 - 2x + 1}{x^2 + 10x + 7}$

$\to \dfrac{3x^4}{x^2} = \boxed{3x^2}$ which is the right end and the left end behavior model

13. $f(x) = 2^{-x} + x^5$

(a) right end, $x \to \infty$, $\boxed{x^5}$

(b) left end, $x \to -\infty$, $\boxed{2^{-x}}$

14. $f(x) = \log_2 |x| + \sqrt[3]{x}$

(a) right end, $x \to \infty$ $\boxed{x^{\frac{1}{3}}}$

(b) left end, $x \to -\infty$ $\boxed{x^{\frac{1}{3}}}$

15. $f(x) = \cos(2x) + x$

(a) right end, $x \to \infty$, $\boxed{x}$

(b) left end, $x \to -\infty$, $\boxed{x}$

16. $f(2) = 1$, $\lim_{x \to \infty} f(x) = -1$, $\lim_{x \to -\infty} f(x) = 1$, $\lim_{x \to 2^+} f(x) = \infty$, $\lim_{x \to 2^-} f(x) = 0$, $\lim_{x \to -1} f(x) = -\infty$