2.4 Rates of Change and Tangent Lines

Slope of a Curve at a Point

The slope of the curve $y = f(x)$ at the point $(a, f(a))$ is the number $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided that the limit exists.

Normal Line

The normal line to a curve at a point is the line perpendicular to the tangent at that point.

Vertical Tangent

The curve $y = f(x)$ has a vertical tangent at $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \infty$ or if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = -\infty$

For problems 1 and 2, find the average rate of change of the function over each interval.

1. $f(x) = x^2 - 3x$ (a) $[-1, 2]$ (b) $[0, 4]$

(a) $\frac{f(2) - f(-1)}{2 - (-1)} = \frac{-2 - 4}{3} = \boxed{-2}$

(b) $\frac{f(4) - f(0)}{4 - 0} = \frac{4 - 0}{4 - 0} = \boxed{1}$

2. $f(x) = \sin\left(\frac{x}{3}\right)$ (a) $[-2\pi, \pi]$ (b) $\left[\frac{-\pi}{2}, \frac{3\pi}{4}\right]$

(a) $\frac{f(\pi) - f(-2\pi)}{\pi - (-2\pi)} = \frac{\frac{\sqrt{3}}{2} - \left(\frac{-\sqrt{3}}{2}\right)}{3\pi} = \boxed{\frac{\sqrt{3}}{3\pi}}$

(b) $\frac{f\left(\frac{3\pi}{4}\right) - f\left(\frac{-\pi}{2}\right)}{\frac{3\pi}{4} - \left(\frac{-\pi}{2}\right)} = \frac{\frac{\sqrt{2}}{2} - \left(\frac{-1}{2}\right)}{\frac{5\pi}{4}} = \frac{4\left(\sqrt{2} + 1\right)}{2(5\pi)}$

$= \boxed{\frac{2\left(\sqrt{2} + 1\right)}{5\pi}}$

For problems 3 – 6, for the indicated point find (a) the slope of the curve (b) the equation of the tangent line (c) the equation of the normal line

3. $f(x) = x^2 + x$ at $x = 1$

(a) $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$= \lim_{h \to 0} \frac{(1+h)^2 + (1+h) - 2}{h}$

$= \lim_{h \to 0} \frac{1 + 2h + h^2 + 1 + h - 2}{h}$

$= \lim_{h \to 0} \frac{3h + h^2}{h} = \lim_{h \to 0} (3 + h) = \boxed{3}$

(b) $\boxed{y - 2 = 3(x - 1)}$ (c) $\boxed{y - 2 = \frac{-1}{3}(x - 1)}$

4. $f(x) = |x + 3|$ at $x = -1$

(a) the slope to the right of $x = -3$ is always 1, so

$\boxed{m = 1}$

(b) $\boxed{y - 2 = 1(x + 1)}$ (c) $\boxed{y - 2 = -1(x + 1)}$

5. $f(x) = \frac{2}{x+1}$ at $x = -2$

(a) $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{\frac{2}{-2+h+1} - (-2)}{h} = \lim_{h \to 0} \frac{\frac{2}{h-1} + 2}{h}$$

$$= \lim_{h \to 0} \frac{2 + 2(h-1)}{h(h-1)} = \lim_{h \to 0} \frac{2h}{h(h-1)}$$

$$= \lim_{h \to 0} \frac{2}{(h-1)} = \boxed{-2}$$

(b) $\boxed{y + 2 = -2(x+2)}$ (c) $\boxed{y + 2 = \frac{1}{2}(x+2)}$

6. $f(x) = x^3$ at $x = \frac{1}{2}$

(a) $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{2} + h\right)^3 - \frac{1}{8}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{8} + \frac{3}{4}h + \frac{3}{2}h^2 + h^3 - \frac{1}{8}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3}{4}h + \frac{3}{2}h^2 + h^3}{h} = \lim_{h \to 0} \left(\frac{3}{4} + \frac{3}{2}h + h^2\right)$$

$$= \boxed{\frac{3}{4}}$$

(b) $\boxed{y - \frac{1}{8} = \frac{3}{4}\left(x - \frac{1}{2}\right)}$ (c) $\boxed{y - \frac{1}{8} = \frac{-4}{3}\left(x - \frac{1}{2}\right)}$

For problems 7 and 8, find the slope of the curve at $x = a$.

7. $f(x) = x^2 + x$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)^2 + (a+h) - (a^2 + a)}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 + (a+h) - (a^2 + a)}{h}$$

$$= \lim_{h \to 0} \frac{2ah + h^2 + h}{h} = \lim_{h \to 0} (2a + h + 1) = \boxed{2a + 1}$$

8. $f(x) = \frac{2}{x+1}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{a+h+1} - \frac{2}{a+1}}{h}$$

$$= \lim_{h \to 0} \frac{2(a+1) - 2(a+h+1)}{h(a+h+1)(a+1)}$$

$$= \lim_{h \to 0} \frac{2a + 2 - 2a - 2h - 2}{h(a+h+1)(a+1)}$$

$$= \lim_{h \to 0} \frac{-2h}{h(a+h+1)(a+1)}$$

$$= \lim_{h \to 0} \frac{-2}{(a+h+1)(a+1)} = \boxed{\frac{-2}{(a+1)^2}}$$

For problems 9 and 10, determine whether the curve has a tangent at the indicated point.

9. $f(x) = 2 - x^2, \quad x \le 1 \quad$ at $x = 1$

$\quad 2 - x, \quad x > 1$

Check the limit from both sides, and

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2 - (1+h)^2 - 1}{h}$$

$$= \lim_{h \to 0^-} \frac{2 - (1 + 2h + h^2) - 1}{h} = \lim_{h \to 0^-} \frac{-2h - h^2}{h}$$

$$= \lim_{h \to 0^-} -2 - h = \boxed{-2 \text{ (from the left),}}$$ and now

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2 - (1+h) - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{-h}{h} = \boxed{-1 \text{ (from the right),}}$$ so,

since $-2 \ne -1$, the curve has $\boxed{\text{no tangent at } x = 1}$

10. $f(x) = \frac{1}{2}x^2 + x, \quad x < 2 \quad$ at $x = 2$

$\quad 3x - 2, \quad x \ge 2$

Check the limit from both sides, and

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0^-} \frac{\frac{1}{2}(2+h)^2 + (2+h) - 4}{h}$$

$$= \lim_{h \to 0^-} \frac{2 + 2h + \frac{h^2}{2} + (2+h) - 4}{h}$$

$$= \lim_{h \to 0^-} \frac{3h + \frac{h^2}{2}}{h} = \lim_{h \to 0^-} 3 + \frac{h}{2} = \boxed{3,}$$ and

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{3(2+h) - 2 - 4}{h}$$

$$= \lim_{h \to 0^+} \frac{3h}{h} = \boxed{3,}$$ so, since $3 = 3$,

the curve has $\boxed{\text{a tangent at } x = 2}$

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11. Find the rate of change of the surface area of a cube with respect to the length of a side, when the length of the side is $s = 3$ inches

First, use the surface area formula for a cube $\rightarrow$ $S = 6x^2$, where x is the length of a side. So the rate of change is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \rightarrow r = \lim_{h \to 0} \frac{6(3+h)^2 - 6(3^2)}{h} = \lim_{h \to 0} \frac{6(9 + 6h + h^2) - 6(3^2)}{h} = \lim_{h \to 0} \frac{36h + 6h^2}{h}$$

$$= \lim_{h \to 0} 36 + 6h = \boxed{36 \, \frac{\text{units of surface area}}{\text{units of length}}}$$

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12. A TI – 89 is dropped from the second floor of the D building. If the height of the calculator is described by $h(t) = 6 - 4.9t^2$ meters, find the velocity of the calculator 1 second after it is dropped

$$v(1) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{6 - 4.9(1+h)^2 - (6 - 4.9)}{h} = \lim_{h \to 0} \frac{6 - 4.9(1 + 2h + h^2) - (6 - 4.9)}{h}$$

$$= \lim_{h \to 0} \frac{-9.8h - 4.9h^2}{h} = \lim_{h \to 0} (-9.8 - 4.9h) = \boxed{-9.8 \, \frac{\text{meters}}{\text{sec}^2}}$$

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13. At what point is the tangent to $f(x) = 4 + 6x + x^2$ horizontal?

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4 + 6(a+h) + (a+h)^2 - (4 + 6a + a^2)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 6a + 6h + a^2 + 2ah + h^2 - (4 + 6a + a^2)}{h} = \lim_{h \to 0} \frac{6h + 2ah + h^2}{h} = \lim_{h \to 0} (6 + 2a + h) = 2a + 6$$

and if the tangent line is horizontal, then $2a + 6 = 0$ so this occurs when $\boxed{a = -3 \text{ (or } x = -3)}$

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14. Find the equations of all tangent lines that pass through the point (1, 1) for the function $f(x) = 1 - \frac{1}{2}x^2$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\left(1 - \frac{1}{2}(a+h)^2\right) - \left(1 - \frac{1}{2}a^2\right)}{h} = \lim_{h \to 0} \frac{\left(1 - \frac{a^2}{2} - ah - \frac{h^2}{2}\right) - \left(1 - \frac{1}{2}a^2\right)}{h}$$

$$= \lim_{h \to 0} \frac{-ah - \frac{h^2}{2}}{h} = \lim_{h \to 0} \frac{-ah - \frac{h^2}{2}}{h} = \lim_{h \to 0} -a - \frac{h}{2} = -a \text{ or } -x,$$ so the equation of the tangent line is

$$y - 1 = -x(x-1) \rightarrow \left(1 - \frac{1}{2}x^2\right) - 1 = -x(x-1) \rightarrow \frac{-1}{2}x^2 = -x^2 + x \rightarrow \frac{1}{2}x^2 - x = 0$$

$x\left(\frac{1}{2}x - 1\right) = 0 \rightarrow x = 0, 2 \rightarrow$ so the points are (0, 1), with slope 0, and (2, –1) with slope –2

and the equations of the tangent lines are $\boxed{y - 1 = 0(x - 0) \text{ and } y + 1 = -2(x - 2)}$