3.1 Derivative of a Function

Derivative

The derivative of the function f with respect to the variable x is the function f', whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{provided the limit exists.}$$

Derivative (Alternate Definition)

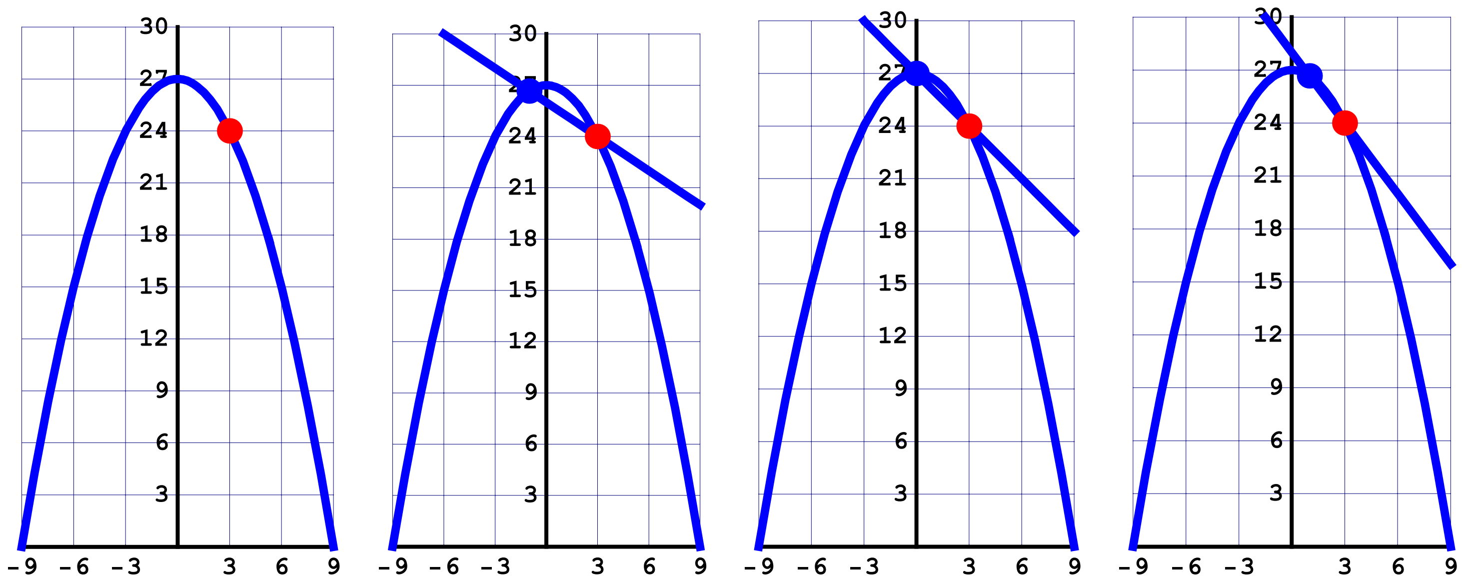
The derivative of the function f at the point $x = a$ is the limit $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, provided the limit exists.

Differentiability on [a, b]

The function $y = f(x)$ is differentiable on a closed interval $[a, b]$ if it has a derivative at every interior point of the interval, and if the limits $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ (the right – hand derivative at a), and

$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$ (the left – hand derivative at b) exist at the endpoints.

The Paolini Building, $f(x) = 27 - \frac{x^2}{3}$



If we want to find the slope of the tangent line at the point $(3, 24)$, we could start with an approximation of a secant line that runs through the curve at the points $\left(-1, 26\frac{2}{3}\right)$ and $(3, 24)$ so $\text{Slope}_1 = \frac{f(3) - f(-1)}{3 - (-1)} = \frac{24 - 26\frac{2}{3}}{4} = \boxed{\frac{-2}{3}}$,

then, for the next approximation, create another secant line through the points $(0, 27)$ and $(3, 24)$, so that

$\text{Slope}_2 = \frac{f(3) - f(0)}{3 - (0)} = \frac{24 - 27}{3} = \boxed{-1,}$ and, for the last approximation, $\text{Slope}_3 = \frac{f(3) - f(2)}{3 - (2)} = \frac{24 - 25\frac{2}{3}}{1} = \boxed{\frac{-5}{3}},$

and if we wish to find the exact slope, we may use a limit, so that $\text{Slope} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{(3+h) - 3}$ so

$$\text{Slope} = \lim_{h \to 0} \frac{27 - \frac{1}{3}(9 + 6h + h^2) - 24}{h} \quad \text{so} \quad \text{Slope} = \lim_{h \to 0} \frac{-2h - \frac{1}{3}h^2}{h} = \lim_{h \to 0} \left(-2 - \frac{1}{3}h\right) = \boxed{-2}$$

1. Use the primary definition of the derivative to find the derivative of $f(x) = x^2 + x$ at $x = -2$

$$f'(-2) = \lim_{h \to 0} \frac{(-2+h)^2 + (-2+h) - ((-2)^2 + (-2))}{h}$$

so $$f'(-2) = \lim_{h \to 0} \frac{4 - 4h + h^2 - 2 + h - 2}{h} = \lim_{h \to 0} \frac{-3h + h^2}{h}$$

so $$f'(-2) = \lim_{h \to 0} (-3 + h) = \boxed{-3}$$

2. Use the secondary definition of the derivative to find the derivative of $f(x) = x^2 + x$ at $x = -2$

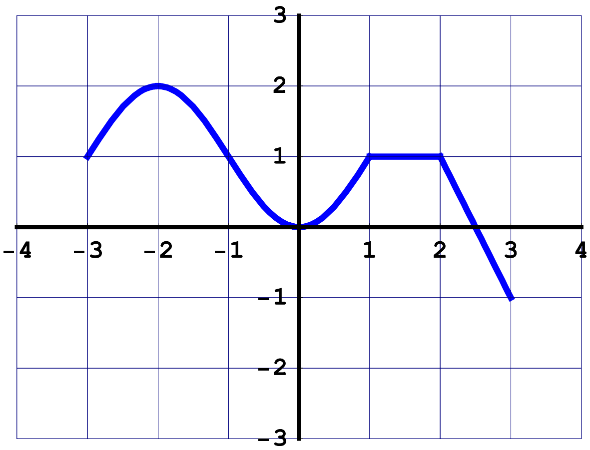
$$f'(-2) = \lim_{x \to -2} \frac{x^2 + x - ((-2)^2 + (-2))}{x - (-2)}$$ so

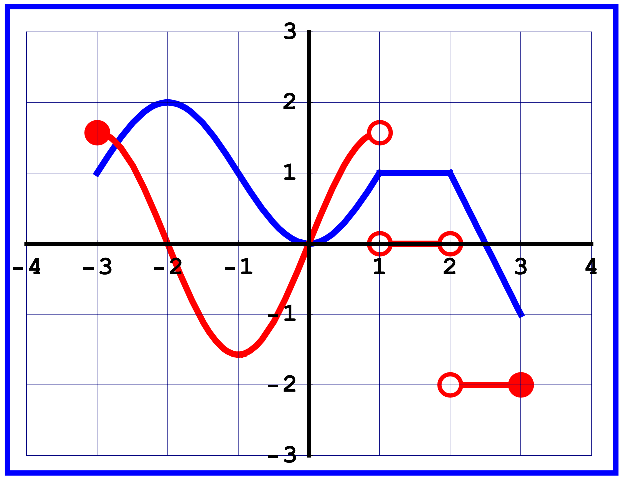
$$f'(-2) = \lim_{x \to -2} \frac{x^2 + x - 2}{x + 2} = \lim_{x \to -2} \frac{(x-1)(x+2)}{x+2}$$

$$f'(-2) = \lim_{x \to -2} (x - 1) = \boxed{-3}$$

For problems 3 – 6, given the graph of $f(x)$, sketch the graph of $f'(x)$.

3.





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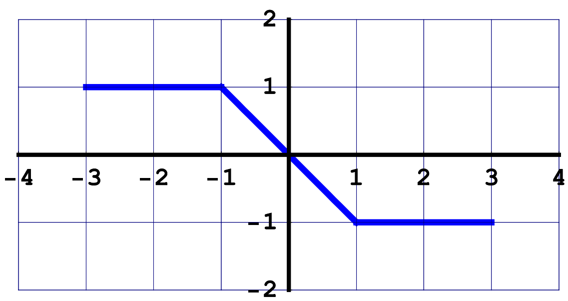

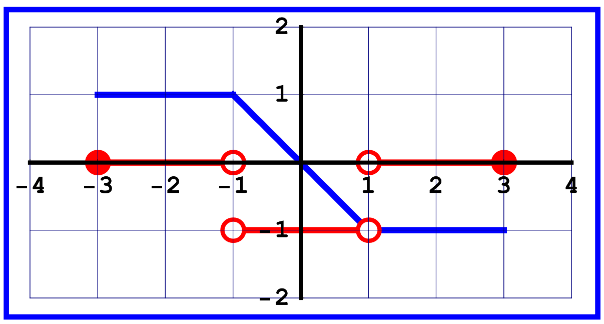

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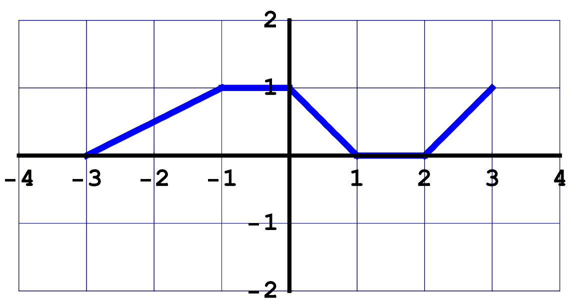

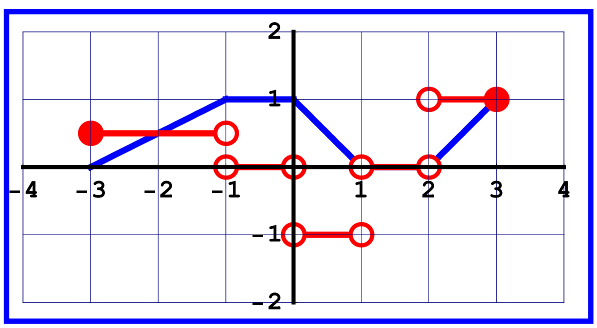

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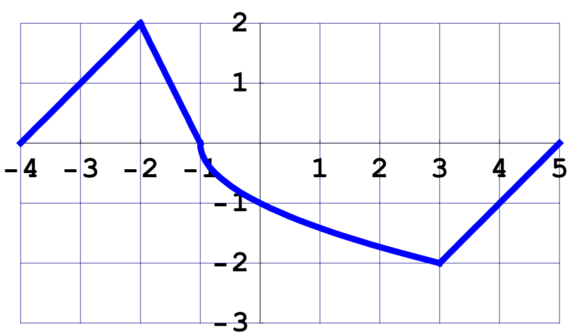

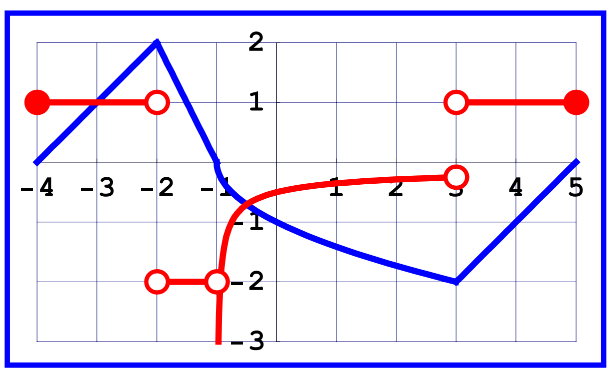

7. For the function $f(x) = \begin{cases} 2x, & x \le -1 \\ -x^2 + x, & x > -1 \end{cases}$ find the left – hand and right – hand derivatives at $x = -1$.

left – hand derivative → $\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0^-} \frac{2(-1+h) - (-2)}{h} = \lim_{h \to 0^-} \frac{-2 + 2h + 2}{h} = \lim_{h \to 0^-} \frac{2h}{h} = \boxed{2}$

right – hand derivative → $\lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \to 0^+} \frac{-(-1+h)^2 + (-1+h) - (-2)}{h}$ so

right – hand derivative $= \lim_{h \to 0^+} \frac{-1 + 2h - h^2 - 1 + h + 2}{h} = \lim_{h \to 0^+} \frac{3h - h^2}{h} = \lim_{h \to 0^-} 3 - h = \boxed{3}$

Since the left and right hand derivatives are different, $\boxed{f'(-1) \text{ Does Not Exist}}$