3.2 Differentiability

Differentiability Implies Continuity

If $f(x)$ has a derivative at $x = a$, then $f(x)$ is continuous at $x = a$.

Intermediate Value Theorem for Derivatives

If a and b are any two $x-$values in an interval in which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$

Left and Right Hand Derivatives

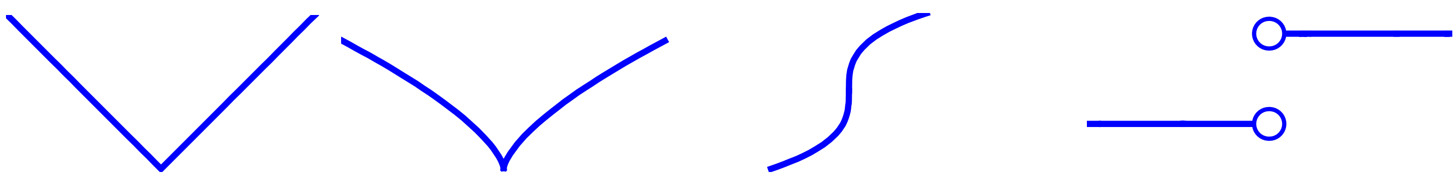
$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Numerical Approximation for Derivatives, Symmetric Difference Quotient

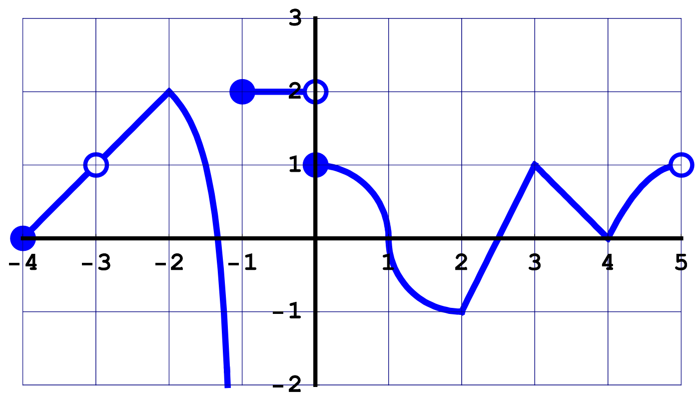
$$\frac{f(a+h) - f(a-h)}{2h}$$

Reasons for derivative failure

A. a corner, $|x|$ B. a cusp, $x^{\frac{2}{3}}$ C. a vertical tangent, $\sqrt[3]{x}$ D. a discontinuity, $\frac{|x|}{x}$



1. Find all points on the graph where the function is not differentiable. Determine why the function is not differentiable at these points.



Not differentiable at the following values

$x = -3$	→	Removable Discontinuity,	$x = -2$	→	Corner
$x = -1$	→	Infinite Discontinuity,	$x = 0$	→	Jump Discontinuity
$x = 1$	→	Vertical Tangent	$x = 2$	→	Corner
$x = 3$	→	Corner	$x = 4$	→	Corner
$x = 5$	→	Can't approach from the left			

For problems 2 – 5, find the left and right hand derivatives to determine if the function is differentiable at the given point.

2. $f(x) = [\![x + 1]\!]$ at $x = 2$

left – hand $\rightarrow \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$

$\rightarrow \lim_{h \to 0^-} \frac{2 - 3}{h} = \boxed{\infty}$ and for the right – hand,

$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{3 - 3}{h} = \boxed{0}$

so the function is **not differentiable at x = 2**

3. $f(x) = \begin{cases} \sin x, & x \neq 0 \\ 1, & x = 0 \end{cases}$ at $x = 0$

left – hand $\rightarrow \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

$\rightarrow \lim_{h \to 0^-} \frac{\sin h - 1}{h} = \lim_{h \to 0^-} \frac{\sin h}{h} + \lim_{h \to 0^-} \frac{-1}{h}$

$= 1 + \infty \rightarrow \boxed{\infty}$ and for the right – hand,

$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{\sin h - 1}{h}$

$= \lim_{h \to 0^+} \frac{\sin h}{h} + \lim_{h \to 0^+} \frac{-1}{h} = 1 - \infty \rightarrow \boxed{-\infty}$

so the function is **not differentiable at x = 0**

4. $f(x) = |x - 2|$ at $x = 2$

left – hand $\rightarrow \lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0^-} \frac{|2 + h - 2| - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \boxed{-1}$

and for the right – hand $\rightarrow \lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0^+} \frac{|2 + h - 2| - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = \boxed{1}$

so the function is **not differentiable at x = 2**

5. $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$ at $x = 1$

left – hand $\rightarrow \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$

$= \lim_{h \to 0^-} \frac{(1+h)^2 - 1 - ((1)^2 - 1)}{h}$

$= \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \boxed{2}$

and for the right – hand $\rightarrow$ there is a jump

discontinuity at x = 1, making $\lim_{h \to 0^+} = \boxed{\infty}$

so the function is **not differentiable at x = 1**

For problems 6 and 7, find all values for which the function is differentiable

6. $f(x) = \frac{x^3 + 1}{x^2 - 2x - 3}$

$f(x) = \frac{(x+1)(x^2 - x + 1)}{(x - 3)(x + 1)}$

there is a removable discontiuity at $x = -1$, and an infinite discontinuity at $x = 3$, so $f(x)$ is differentiable on the interval $\boxed{(-\infty, -1) \cup (-1, 3) \cup (3, \infty)}$

7. $f(x) = \log_3 |x|$

Looking at the graph, there is an infinite

disontinuity at $x = 0$, so $f(x)$ is differentiable on the interval $\boxed{(-\infty, 0) \cup (0, \infty)}$