3.6 Chain Rule

Chain Rule

$D_x[f(g(x))] = f'(g(x))\,g'(x)$ or if $y = f(u)$, $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Derivative of a Parametrically Defined Plane Curve

For a parametrically defined plane curve, if $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$

For problems 1 – 8, find $\dfrac{dy}{dx}$.

1. $y = \tan u, \quad u = 3x^2$

$y = \tan(3x^2) \qquad \dfrac{dy}{dx} = \sec^2(3x^2)(6x)$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \sec^2 u\,(6x) = \boxed{\sec^2(3x^2)(6x)}$

2. $y = \sqrt[3]{u}, \quad u = x^2 + 5x$

$y = \sqrt[3]{x^2 + 5x} \qquad \dfrac{dy}{dx} = \dfrac{1}{3}(x^2 + 5x)^{-\frac{2}{3}}(2x + 5)$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{3}u^{-\frac{2}{3}}(2x + 5) = \boxed{\dfrac{1}{3}(x^2 + 5x)^{-\frac{2}{3}}(2x + 5)}$

3. $y = -2x\cot\left(\dfrac{\sqrt{x}}{3}\right)$

$\dfrac{dy}{dx} = \boxed{(-2)\cot\left(\dfrac{\sqrt{x}}{3}\right) + (-2x)\left(-\csc^2\left(\dfrac{\sqrt{x}}{3}\right)\left(\dfrac{1}{6}x^{-\frac{1}{2}}\right)\right)}$

4. $y = (3x^2 - x + 2)(x^3 + 5x)^4$

$\dfrac{dy}{dx} = \boxed{(6x - 1)(x^3 + 5x)^4 + (3x^2 - x + 2)(4)(x^3 + 5x)^3(3x^2 + 5)}$

5. $y = \csc(\cot^2 x)$

$y = \csc(u) \qquad u = w^2 \qquad w = \cot x$

$\dfrac{dy}{du} = -\csc(u)\cot(u) \qquad \dfrac{du}{dw} = 2w \qquad \dfrac{dw}{dx} = -\csc^2 x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dw}\dfrac{dw}{dx}$, so

$\dfrac{dy}{dx} = \boxed{-\csc(\cot^2(x))\cot(\cot^2(x))(2\cot(x))(-\csc^2 x)}$

6. $y = \left(\dfrac{3x + 4}{6x - 1}\right)^3$

$\dfrac{dy}{dx} = \boxed{3\left(\dfrac{3x + 4}{6x - 1}\right)^2\left(\dfrac{(6x - 1)(3) - (3x + 4)(6)}{(6x - 1)^2}\right)}$

7. $y = \sin^3(2x)\sqrt{\cos(3x)}$

$$\dfrac{dy}{dx} = \boxed{\begin{array}{l} 3(\sin^2(2x))\cos(2x)\,2\sqrt{\cos(3x)} \\ +\ \sin^3(2x)\,\dfrac{1}{2}(\cos(3x))^{-\frac{1}{2}}(-\sin(3x))(3) \end{array}}$$

8. $y = \cos^2\sqrt{3 - 7x}$

$\dfrac{dy}{dx} = \boxed{2\cos\sqrt{3 - 7x}\left(-\sin\sqrt{3 - 7x}\right)\left(\dfrac{1}{2}(3 - 7x)^{-\frac{1}{2}}\right)(-7)}$

For problems 9 – 12, find the equation of the tangent line at the given value of t.

9. $x = \sec t, \quad y = \tan t, \quad t = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \frac{1}{\cos t}\frac{\cos t}{\sin t} = \csc t$$

$\csc\frac{\pi}{4} = \sqrt{2}$, $\quad x = \sec\frac{\pi}{4} = \sqrt{2}$, $\quad y = \tan\frac{\pi}{4} = 1$

$$\boxed{y - 1 = \sqrt{2}\left(x - \sqrt{2}\right)}$$

10. $x = 3t^2 - 6t, \quad y = \sqrt{t}, \quad t = 4$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{2}(t)^{-\frac{1}{2}}}{6t - 6} = \frac{1}{12(t-1)\left(t^{\frac{1}{2}}\right)}$$

$$y'(4) = \frac{1}{12(4-1)\left(4^{\frac{1}{2}}\right)} = \frac{1}{72}$$

$x = 3(4)^2 - 6(4) = 24 \qquad y = \sqrt{4} = 2$

$$\boxed{y - 2 = \frac{1}{72}(x - 24)}$$

11. $x = \cos^3 t, \quad y = \sin^2 t, \quad t = \frac{-\pi}{6}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\sin t(\cos t)}{3\cos^2 t(-\sin t)} = \frac{-2}{3}\sec t$$

$$\frac{dy}{dx} = \left(\frac{-2}{3}\right)\left(\frac{2}{\sqrt{3}}\right) = \frac{-4\sqrt{3}}{9} \qquad x = \frac{3\sqrt{3}}{8}, \qquad y = \frac{1}{4}$$

$$\boxed{y - \frac{1}{4} = \frac{-4\sqrt{3}}{9}\left(x - \frac{3\sqrt{3}}{8}\right)}$$

12. $x = t^3 + 1, \quad y = t^2 - 2t, \quad t = -2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 2}{3t^2} = \frac{2(t-1)}{3t^2}$$

$$\frac{dy}{dx} = \frac{2(-2-1)}{3(-2)^2} = \frac{-6}{12} = \frac{-1}{2}$$

$x = (-2)^3 + 1 = -7 \qquad y = (-2)^2 - 2(-2) = 8$

$$\boxed{y - 8 = \frac{-1}{2}(x + 7)}$$

13. Let $s = \csc t$. Find $\frac{ds}{dy}$ when $t = \frac{2\pi}{3}$ and $\frac{dt}{dy} = \frac{1}{2}$.

$$\frac{ds}{dy} = \frac{ds}{dt}\frac{dt}{dy}$$

$$\frac{ds}{dt} = -\csc(t)\cot(t) \qquad \frac{dt}{dy} = \frac{1}{2}$$

$$\frac{ds}{dy} = \frac{1}{2}(-\csc(t)\cot(t)) = \frac{1}{2}\left(-\csc\left(\frac{2\pi}{3}\right)\cot\left(\frac{2\pi}{3}\right)\right) = \frac{1}{2}\left(-\left(\frac{2\sqrt{3}}{3}\right)\left(\frac{-\sqrt{3}}{3}\right)\right) = \boxed{\frac{1}{3}}$$

For problems 14 – 19, compare $\lim\limits_{x \to a^+} f'(x)$ and $\lim\limits_{x \to a^-} f'(x)$ and $\lim\limits_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ and $\lim\limits_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$

14. $f(x) = 1, \ x \neq 1$

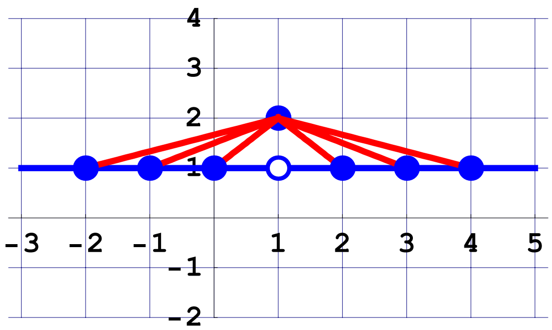
$\qquad 2, \ x = 1 \qquad a = 1$

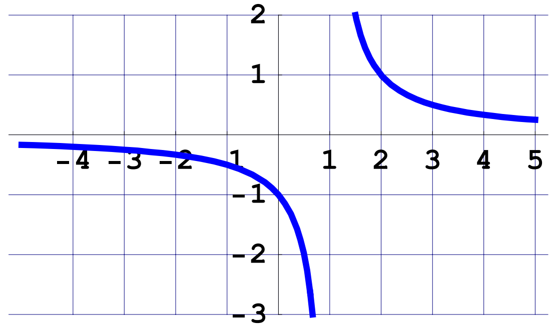
$$\lim_{x \to 1^+} f'(x) = 0$$
$$\lim_{x \to 1^-} f'(x) = 0$$
$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = -\infty$$
$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \infty$$

15. $f(x) = \frac{1}{x-1} \qquad a = 1$

$$\lim_{x \to 1^+} f'(x) = -\infty$$
$$\lim_{x \to 1^-} f'(x) = -\infty$$
$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \text{DNE}$$
$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \text{DNE}$$

(no anchor point)





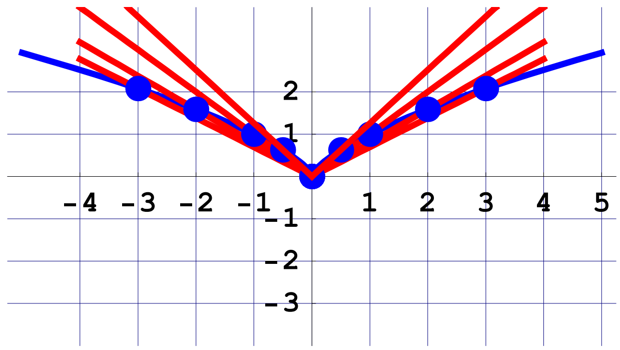
16. $f(x) = x^{\frac{2}{3}}$ $a = 0$

$$\lim_{x \to 0^+} f'(x) = \infty$$

$$\lim_{x \to 0^-} f'(x) = -\infty$$

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \infty$$

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = -\infty$$



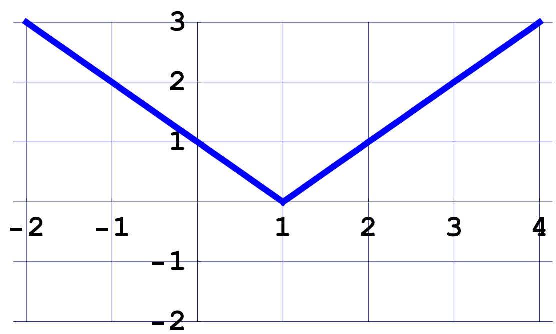
17. $f(x) = |x - 1|$ $a = 1$

$$\lim_{x \to 1^+} f'(x) = 1$$

$$\lim_{x \to 1^-} f'(x) = -1$$

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = -1$$



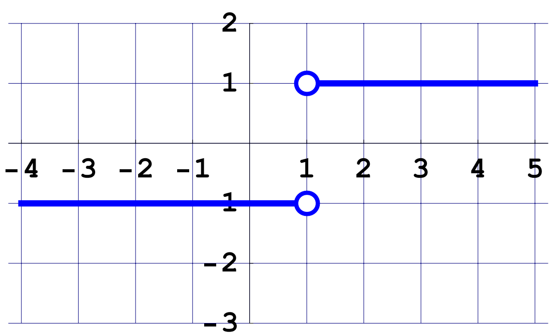
18. $f(x) = \dfrac{|x - 1|}{x - 1}$ $a = 1$

$$\lim_{x \to 1^+} f'(x) = 0$$

$$\lim_{x \to 1^-} f'(x) = 0$$

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = \text{DNE}$$ (no anchor point)

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \text{DNE}$$



19. $f(x) = 2x, \quad x < 1$

$x^2, \quad x \geq 1, \quad a = 1$

$$\lim_{x \to 1^+} f'(x) = 2$$

$$\lim_{x \to 1^-} f'(x) = 2$$

$$\lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h} = 2$$

$$\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = -\infty$$