3.7 Implicit Differentiation

Implicit Differentiation Process

(1) Differentiate both sides of the equation with respect to x

(2) Collect the terms with $\frac{dy}{dx}$ on one side of the equation

(3) Factor out $\frac{dy}{dx}$

(4) Solve for $\frac{dy}{dx}$

For problems 1 – 5, assume that the equation determines a differentiable function f such that $y = f(x)$, and find y'

1. $y = 5\sqrt{3x^{\frac{-1}{3}} + 2}$

$$\frac{dy}{dx} = \frac{5}{2}\left(3x^{\frac{-1}{3}} + 2\right)^{\frac{-1}{2}}\left(-1x^{\frac{-4}{3}} + 0\right)$$

2. $2x - \sqrt{xy} + y^3 = 16$

Letting $y = f(x) \rightarrow 2x - \sqrt{x\,f(x)} + (f(x))^3 = 16$

$$2 - \frac{1}{2}(x\,f(x))^{\frac{-1}{2}}(f(x) + x\,f'(x)) + 3(f(x))^2 f'(x) = 0$$

$$2 - \frac{1}{2}x^{\frac{-1}{2}}(f(x))^{\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}(f(x))^{\frac{-1}{2}}f'(x) + 3(f(x))^2 f'(x) = 0$$

$$2 - \frac{1}{2}x^{\frac{-1}{2}}(f(x))^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}}(f(x))^{\frac{-1}{2}}f'(x) - 3(f(x))^2 f'(x)$$

Therefore, $f'(x) = \dfrac{4 - x^{\frac{-1}{2}}(f(x))^{\frac{1}{2}}}{x^{\frac{1}{2}}(f(x))^{\frac{-1}{2}} - 6(f(x))^2} = \dfrac{4 - \frac{\sqrt{y}}{\sqrt{x}}}{\frac{\sqrt{x}}{\sqrt{y}} - 6y^2}$

3. $\sin^2(3y) = x + y - 1$

$2(\sin(3y))\cos(3y)3y' = 1 + y'$

$1 = 2(\sin(3y))\cos(3y)3y' - y'$

$$\frac{dy}{dx} = \frac{1}{2\sin(3y)\cos(3y)3 - 1}$$

4. $y^2 + 1 = x^2 \sec y$

$2yy' = 2x\sec y + x^2 \sec y \tan y\, y'$

$$\frac{dy}{dx} = \frac{2x\sec y}{2y - x^2\sec y\tan y}$$

5. $y = \csc(xy)$

$y' = -(\csc(xy)\cot(xy))(xy' + y)$

$$\frac{dy}{dx} = \frac{-y\csc(xy)\cot(xy)}{1 + x\csc(xy)\cot(xy)}$$

6. Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $3x^2 + 4y^2 = 4$

$6x + 8yy' = 0 \rightarrow y' = \dfrac{-3x}{4y}$

$$y'' = \frac{4y(-3) - (4y')(-3x)}{16y^2} = \frac{4y(-3) - \left(4\left(\frac{-3x}{4y}\right)\right)(-3x)}{16y^2} = \frac{-48y^2 + 36x^2}{64y^3} = \frac{-12y^2 + 9x^2}{16y^3}$$

7. Find the equation of the tangent line for $x^2 y + \sin y = 2\pi$ at the point $(1, 2\pi)$

$2xy + x^2 y' + \cos y\, y' = 0 \quad \rightarrow \quad y' = \dfrac{-2xy}{x^2 + \cos y} \quad \rightarrow \quad y'(1, 2\pi) = \dfrac{-4\pi}{2} = -2\pi$

$\boxed{y - 2\pi = -2\pi(x - 1)}$

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8. Find the equation of the normal line for $(x^2 + y^2)^2 = 4xy$ at the point $(1, 1)$

$2(x^2 + y^2)(2x + 2yy') = 4y + 4xy' \quad \rightarrow \quad 4(2 + 2y') = 4 + 4y' \quad \rightarrow \quad 4y' = -4 \quad \rightarrow \quad y' = -1$

$m = 1 \quad \rightarrow \quad \boxed{y - 1 = 1(x - 1)}$