3.9 Derivatives of Exponential and Logarithmic Functions

Exponential and Logarithmic Function Derivatives

$D_x e^u = e^u \dfrac{du}{dx}$ $\qquad$ $D_x a^u = a^u \ln a \dfrac{du}{dx}$

$D_x \ln u = \dfrac{1}{u} \dfrac{du}{dx}$ $\qquad$ $D_x \log_a u = \dfrac{1}{u \ln a} \dfrac{du}{dx}$

Logarithmic Differentiation

(1) Take the natural log of both sides of the equation

(2) Differentiate both sides

(3) Isolate $\dfrac{dy}{dx}$

For problems 1 – 10, find the derivative.

1. $y = e^{\frac{-x}{2}}$

$$y' = \frac{-1}{2}\left(e^{\frac{-x}{2}}\right)$$

2. $y = x^3 e^x + \sqrt{x}\, e^{2x}$

$$\frac{dy}{dx} = 3x^2 e^x + x^3 e^x + \frac{1}{2} x^{\frac{-1}{2}} e^{2x} + \sqrt{x}\,(e^{2x}(2))$$

3. $y = x^{3e+2}$

$$D_x y = (3e + 2)\, x^{3e+1}$$

4. $y = 4^{-x}$

$$D_x y = 4^{-x} (\ln 4)(-1)$$

5. $y = 5^{\sin x}$

$$\frac{dy}{dx} = 5^{\sin x} (\ln 5)(\cos x)$$

6. $y = \ln\left(\dfrac{5}{x^2}\right)$

$$y' = \frac{1}{5x^{-2}}(-10x^{-3})$$

7. $y = \ln(5 + \cos x)$

$$\frac{dy}{dx} = \frac{1}{5 + \cos x}(-\sin x)$$

8. $y = \ln(\log_5 x^2)$

$$\frac{dy}{dx} = \frac{1}{2\log_5 x} \frac{2}{x \ln 5}$$

9. $y = \log_3 (e^x)$

$$\frac{dy}{dx} = \frac{1}{e^x} \frac{1}{\ln 3} (e^x) = \frac{1}{\ln 3} = \frac{1}{\log_e 3} = \frac{1}{\frac{\log_3 3}{\log_3 e}} = \log_3 e$$

$$D_x y = \log_3 e$$

10. $y = \dfrac{1}{\log_6 (x - 2)}$

$$D_x y = -(\log_6 (x-2))^{-2}\left(\frac{1}{\ln 2}\right)\left(\frac{1}{x-2}\right)(1)$$

For problems 11 and 12, find the derivative using logarithmic differentiation.

11. $y = (\cos x)^x$

$\ln y = \ln(\cos x)^x \quad \rightarrow \quad \ln y = x \ln(\cos x)$

$$\frac{1}{y} y' = (1) \ln(\cos x) + x\left(\frac{1}{\cos x}\right)(-\sin x)$$

$$y' = y(\ln(\cos x) - x \tan x)$$

$$y' = ((\cos x)^x)(\ln(\cos x) - x \tan x)$$

12. $y = \dfrac{x^2 \sqrt{4x - 3}}{(x^2 + 1)^{\frac{3}{4}}}$

$$\ln y = 2\ln x + \frac{1}{2}\ln(4x - 3) - \frac{3}{4}\ln(x^2 + 1)$$

$$\frac{1}{y} y' = \frac{2}{x} + \frac{1}{2}\frac{1}{4x - 3}(4) - \frac{3}{4}\frac{1}{x^2 + 1}(2x)$$

$$\frac{dy}{dx} = \frac{x^2\sqrt{4x - 3}}{(x^2 + 1)^{\frac{3}{4}}}\left(\frac{2}{x} + \frac{2}{4x - 3} - \frac{3x}{2(x^2 + 1)}\right)$$

13. Find the equations of the tangent and normal lines to the curve $y^2(2-x) = x^3$ at the point $(1, 1)$.

$$2yy'(2-x) - y^2 = 3x^2 \quad \rightarrow \quad 2y' - 1 = 3 \quad y' = 2$$

Tangent Line $\rightarrow$ $y - 1 = 2(x-1)$

Normal Line $\rightarrow$ $y - 1 = \frac{-1}{2}(x-1)$

14. Find points on the curve $x^2 + xy + y^2 = 7$ where the tangent line is horizontal, and where the tangent line is vertical.

$$2x + (1)y + xy' + 2yy' = 0 \qquad y'(x+2y) = -2x - y$$

$\frac{dy}{dx} = \frac{-2x-y}{x+2y}$ Tangent Line is horizontal $\rightarrow$ $y = -2x$

$x^2 - 2x^2 + 4x^2 = 7 \qquad x^2 = \frac{7}{3} \qquad \rightarrow \quad x = \frac{\pm\sqrt{21}}{3}$

Tangent Line is vertical $\rightarrow$ $y = \frac{-1}{2}x \qquad x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2 = 7$

$x^2 = \frac{28}{3} \qquad x = \frac{\pm 2\sqrt{21}}{3}$