4.1 Extreme Values of Functions

Absolute (or Global) Extreme Values

Let f be a function with domain D. Then $f(c)$ is the

(a) absolute maximum on D if and only if $f(x) \le f(c)$ for all x in D

(b) absolute minimum on D if and only if $f(x) \ge f(c)$ for all x in D

Extreme Value Theorem

If f is continuous on $[a, b]$, then f has a maximum value and a minimum value on the interval.

Local (or Relative) Extreme Values

Let c be an interior point of the domain of the function of f. Then $f(c)$ is a

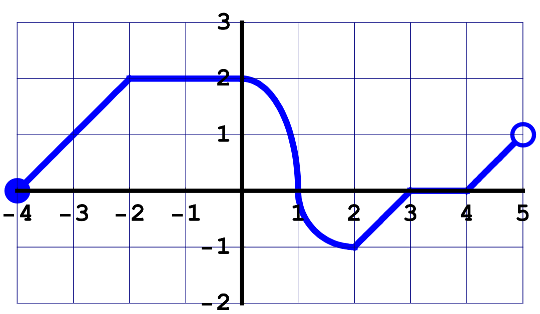
(a) local maximum at c if and only if $f(x) \le f(c)$ for all x in some open interval containing c

(b) local minimum at c if and only if $f(x) \ge f(c)$ for all x in some open interval containing c

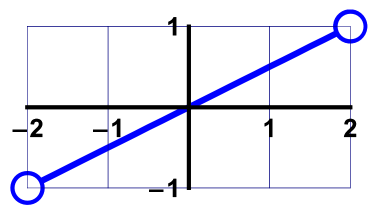
Critical Point

A point $(c, f(c))$ in the interior of the domain of a function f at which $f'(c) = 0$ or $f'(c)$ does not exist is a critical point of f.

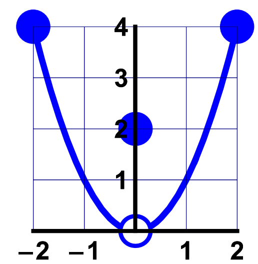
1. for the following functions, find all local and absolute extreme values.



Absolute minimum at $x = 2$, is -1
or the point $(2, -1)$
Absolute maximum on $[-2, 0]$ is 2
Local minimum values at the point $(-4, 0)$
and at the point $(2, -1)$
Local minimums on $(3, 4]$ is 0
also on $(-2, 0)$ is 2
Local maximum values on $[3, 4)$ is 0
also on $[-2, 0]$ is 2

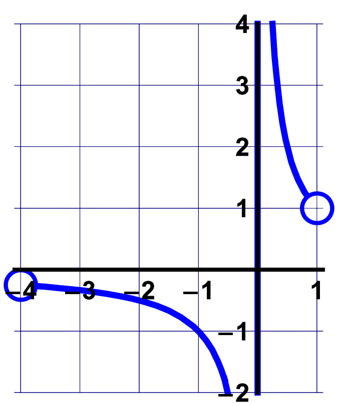


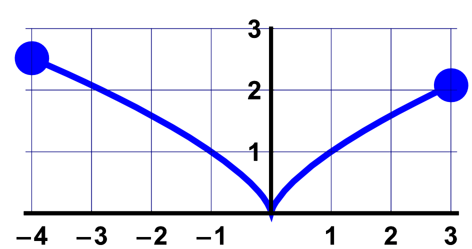
no absolute extreme values,
no local extreme values



Absolute maximum → $(-2, 4)$ and $(2, 4)$
Absolute minimum → none
Local maximum values → $(0, 2)$ $(-2, 4)$,
and $(2, 4)$
Local minimum values → none

For problems 2 – 9, find the extreme values of the function, and where they occur.





2. $f(x) = \frac{1}{x}$ on $(-4, 1)$

$f(x) = x^{-1}$ $f'(x) = \frac{-1}{x^2}$

critical values → none (0 not in domain), and

no endpoints, so no local and no absolute extreme values

3. $f(x) = \sqrt[3]{x^2}$ on $[-4, 3]$

$f'(x) = \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3\sqrt[3]{x}}$ critical value → $c = 0$

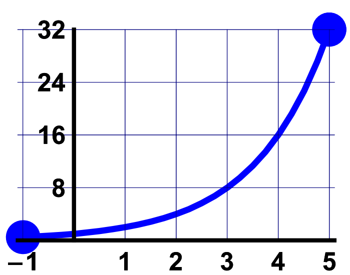
$f(0) = 0$ and now, check endpoints,

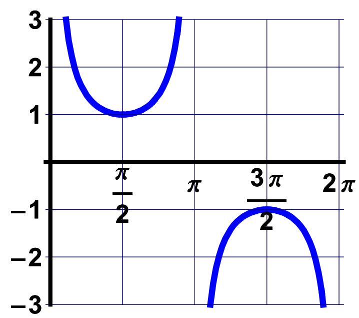
$f(-4) = \sqrt[3]{16}$, $f(3) = \sqrt[3]{9}$ so

local minimum at $(0, 0)$, local maxima at $\left(-4, \sqrt[3]{16}\right)$, $\left(3, \sqrt[3]{9}\right)$

and absolute minimum at $(0, 0)$

absolute maximum at $\left(-4, \sqrt[3]{16}\right)$





4. $f(x) = 2^x$ on $[-1, 5]$

$f'(x) = 2^x(\ln 2)$ so no critical points, and we now

check the endpoints → $f(-1) = \frac{1}{2}$ and $f(5) = 32$

so local minimum at $\left(-1, \frac{1}{2}\right)$ and

local maximum at $(5, 32)$ so

absolute min at $\left(-1, \frac{1}{2}\right)$ and absolute max at $(5, 32)$

5. $f(x) = \csc x$ on $(0, 2\pi)$

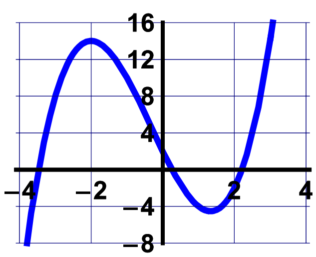
$f'(x) = -\csc x \cot x = \frac{-1}{\sin x}\frac{\cos x}{\sin x} = \frac{-\cos x}{\sin^2 x}$

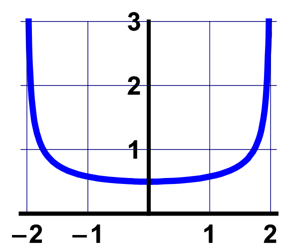
critical values → $c = \frac{\pi}{2}, \frac{3\pi}{2}$ (not π, because not in domain)

$f\left(\frac{\pi}{2}\right) = 1$, $f(\pi)$ → DNE, $f\left(\frac{3\pi}{2}\right) = -1$, and

local minimum at $\left(\frac{\pi}{2}, 1\right)$ and local maximum at $\left(\frac{3\pi}{2}, -1\right)$,

no absolute extreme values





6. $y = x^3 + x^2 - 8x + 2$

Domain: All Reals $\quad y' = 3x^2 + 2x - 8 = 0$

$\rightarrow \quad (3x - 4)(x + 2) = 0,$

$\rightarrow$ critical values $c = \dfrac{4}{3}, -2$

$$f\left(\frac{4}{3}\right) = \frac{64}{27} + \frac{16}{9} - \frac{32}{3} + 2 = \frac{-122}{27}$$

$f(-2) = -8 + 4 + 16 + 2 = 14$ so

local min at $\left(\dfrac{4}{3}, \dfrac{-122}{27}\right)$, and local max at $(-2, 14)$
no global extreme values

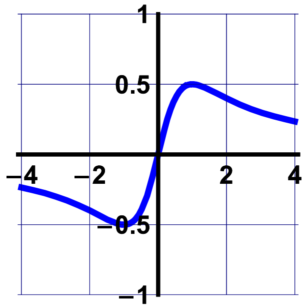
7. $y = \dfrac{1}{\sqrt{4 - x^2}}$

Domain: $(-2, 2)$ $\quad y = (4 - x^2)^{\frac{-1}{2}}$

$$y' = \frac{-1}{2}(4 - x^2)^{\frac{-3}{2}}(-2x) = \frac{x}{\sqrt{(4 - x^2)^3}}$$

$c = 0$ and $f(0) = \dfrac{1}{2}$, and $\displaystyle\lim_{x \to 2^-} \frac{1}{\sqrt{4 - x^2}} \rightarrow \infty$, and

$\displaystyle\lim_{x \to -2^+} \frac{1}{\sqrt{4 - x^2}} \rightarrow \infty$, so local minimum at $\left(0, \dfrac{1}{2}\right)$
and no local maximum, and absolute minimum at $\left(0, \dfrac{1}{2}\right)$
and no absolute maximum



8. $y = \dfrac{x}{x^2 + 1}$

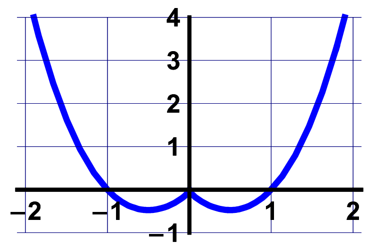
Domain: All Reals, and

$$y' = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

so critical values at $c = -1, 1$ and

$f(-1) = \dfrac{-1}{2}$, $f(1) = \dfrac{1}{2}$ and $\displaystyle\lim_{x \to \infty} \frac{x}{x^2 + 1} = 0,$

$\displaystyle\lim_{x \to -\infty} \frac{x}{x^2 + 1} = 0$ so local minimum at $\left(-1, \dfrac{-1}{2}\right)$
and local maximum at $\left(1, \dfrac{1}{2}\right)$ and
absolute minimum at $\left(-1, \dfrac{-1}{2}\right)$,
absolute maximum at $\left(1, \dfrac{1}{2}\right)$



9. $y = x^{\frac{2}{3}}(x^2 - 1)$

$y = x^{\frac{8}{3}} - x^{\frac{2}{3}}$ $\quad$ Domain: All Reals, $\quad y' = \dfrac{8}{3}x^{\frac{5}{3}} - \dfrac{2}{3}x^{\frac{-1}{3}}$

$$y' = \frac{8x^2 - 2}{3x^{\frac{1}{3}}} = \frac{2(2x - 1)(2x + 1)}{3x^{\frac{1}{3}}}$$ so

critical values $c = \dfrac{1}{2}, \dfrac{-1}{2}, 0$ and

$f\left(\dfrac{-1}{2}\right) = \dfrac{-3}{4\sqrt[3]{4}}$ $\quad f(0) = 0$ $\quad f\left(\dfrac{1}{2}\right) = \dfrac{-3}{4\sqrt[3]{4}}$ so

local minima at $\left(\dfrac{-1}{2}, \dfrac{-3}{4\sqrt[3]{4}}\right)$ and $\left(\dfrac{1}{2}, \dfrac{-3}{4\sqrt[3]{4}}\right)$
local maximum at $(0, 0)$ and
absolute minima at $\left(\dfrac{-1}{2}, \dfrac{-3}{4\sqrt[3]{4}}\right)$ and $\left(\dfrac{1}{2}, \dfrac{-3}{4\sqrt[3]{4}}\right)$
no absolute maximum