4.2 Mean Value Theorem

Mean Value Theorem

If $y = f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one c on (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Increasing and Decreasing Functions

Let f be continuous on $[a, b]$ and differentiable on (a, b)

(1) If $f' > 0$ on (a, b) then f increases on $[a, b]$

(2) If $f' < 0$ on (a, b) then f decreases on $[a, b]$

Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval I, then there is a constant C such that $f(x) = g(x) + C$ for all x in I.

Antiderivative

A function $F(x)$ is an antiderivative of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The process of finding an antiderivative is antidifferentiation.

For problems 1 – 4, find (a) the local extrema, (b) the intervals on which the function is increasing, and (c) the intervals on which the function is decreasing.

1. $f(x) = x^2 + 2x - 15$

Domain : All Reals, and $f'(x) = 2x + 2 = 0$

so, critical value $\rightarrow$ $c = -1$ $\rightarrow$ $f(-1) = -16$

local minimum at the point $(-1, -16)$, no local maximum, and absolute minimum at the point $(-1, -16)$, no absolute maximum

function is increasing on the interval $[-1, \infty)$ and

decreasing on the interval $(-\infty, -1]$

2. $f(x) = \left(\frac{1}{2}\right)^x$

Domain : All Reals, and $f'(x) = -(\ln 2)\, 2^{-x}$

so no critical values, and since the derivative in always negative, the function is

decreasing throughout its domain, and no local nor absolute extreme values

3. $f(x) = \frac{5}{x^4}$

(Keep in mind that the function is an even function)

Domain : $(-\infty, 0) \cup (0, \infty)$, and $f'(x) = \frac{-20}{x^5}$

so no critical values because $f(0)$ DNE

no local maximum or minimum
function is increasing on the interval $(-\infty, 0)$,
and decreasing on $(0, \infty)$

4. $f(x) = \frac{x}{x^2 - 9}$

(Keep in mind that the function is an odd function)

Domain : $(-\infty, 3) \cup (-3, 3) \cup (3, \infty)$, and

$$f'(x) = \frac{(x^2 - 9) - 2x^2}{(x^2 - 9)^2} = \frac{-(x^2 + 9)}{(x^2 - 9)^2}$$

so, no critical values because $f(3)$ and $f(-3)$ DNE

There are no local maxima or minima and function is decreasing on the intervals $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$

For problems 5 – 8, (a) determine whether the function f satisfies the hypotheses of the Mean Value Theorem on the given interval, and (b) find all values of c that satisfy the equation $f'(c) = \frac{f(b) - f(a)}{b - a}$

5. $f(x) = \sqrt[3]{x+2}$ $[-3, 6]$

(a) continuous on $[-3, 6]$ and $f'(x) = \frac{1}{3}(x+2)^{\frac{-2}{3}}$

so it's not differentiable at $x = -2$,

so, **does not satisfy the if clause of the Mean Value Theorem**

(b) $\frac{1}{3}(c+2)^{\frac{-2}{3}} = \frac{f(6) - f(-3)}{6 - (-3)} = \frac{1}{3} \rightarrow (c+2)^{\frac{-2}{3}} = 1$

$\rightarrow$ **$c = -1$** We don't include -3 as a solution because it is an endpoint of our domain

6. $f(x) = \tan^{-1} x$ $[-1, \sqrt{3}]$

(a) **continuous on $[-1, \sqrt{3}]$** and $f'(x) = \frac{1}{1+x^2}$

so it is **differentiable on $(-1, \sqrt{3})$**

(b) $\frac{1}{1+c^2} = \frac{f(\sqrt{3}) - f(-1)}{\sqrt{3} - (-1)} = \frac{\frac{\pi}{3} - \left(\frac{-\pi}{4}\right)}{\sqrt{3} - (-1)} = \frac{\frac{7\pi}{12}}{\sqrt{3} + 1}$

so $1 + c^2 = \frac{12\sqrt{3} + 12}{7\pi}$

$\rightarrow$ **$c = \pm\sqrt{\frac{12\sqrt{3} + 12 - 7\pi}{7\pi}} \approx \pm 0.70058$**

7. $f(x) = x + \frac{4}{x}$ $[-4, 1]$

(a) **discontinuous and not differentiable at $x = 0$**

(b) $1 - \frac{4}{x^2} \rightarrow \frac{c^2 - 4}{c^2} = \frac{f(1) - f(-4)}{1 - (-4)} = \frac{5 - (-5)}{5} = 2$

$\rightarrow c^2 - 4 = 2c^2 \rightarrow$ **$c^2 = -4$, no solution**

8. $f(x) = \cos\frac{x}{2}$ $[\frac{-\pi}{2}, \pi]$

(a) **continuous on $[\frac{-\pi}{2}, \pi]$** and $f'(x) = \frac{-1}{2}\sin\frac{x}{2}$

so it is **differentiable on $(\frac{-\pi}{2}, \pi)$**

(b) $\frac{-1}{2}\sin\frac{c}{2} = \frac{f(\pi) - f\left(\frac{-\pi}{2}\right)}{\pi - \left(\frac{-\pi}{2}\right)} = \frac{0 - \left(\frac{\sqrt{2}}{2}\right)}{\frac{3\pi}{2}} = \frac{-\sqrt{2}}{3\pi}$

$\rightarrow \sin\frac{c}{2} = \frac{2\sqrt{2}}{3\pi}$ so **$c = 2\sin^{-1}\left(\frac{2\sqrt{2}}{3\pi}\right)$**

For problems 9 – 11, find all possible functions f with the given derivative.

9. $f'(x) = x^3 + x^2 - x$

$\rightarrow$ **$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + C$**

where C is any constant

10. $f'(x) = \frac{2}{\sqrt{1 - 4x^2}}$

$\rightarrow$ **$f(x) = \sin^{-1}(2x) + C$**

where C is any constant

11. $f'(x) = (\ln 5)\, 5^x$

$\rightarrow$ **$f(x) = 5^x + C$** where C is any constant