4.3 Connecting f' and f'' with the Graph of f

First Derivative Test

Let $f(x)$ be continuous on a given open interval that contains c, a critical value

(1) If $f'(x)$ changes from $+$ to $-$ at c, then $(c, f(c))$ is a local maximum

(2) If $f'(x)$ changes from $-$ to $+$ at c, then $(c, f(c))$ is a local minimum

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Concavity Test

Let $f(x)$ be continuous on a given interval (a, b)

(1) $f(x)$ is Concave Up if $f''(x) > 0$ on (a, b)

(2) $f(x)$ is Concave Down if $f''(x) < 0$ on (a, b)

Point of Inflection

$(c, f(c))$ is a Point of Inflection if $f'(c)$ exists and $f''(x)$ changes sign at c.

Second Derivative Test

(1) If $f'(c) = 0$ and $f''(c) < 0$, then $(c, f(c))$ is a local maximum

(2) If $f'(c) = 0$ and $f''(c) > 0$, then $(c, f(c))$ is a local minimum

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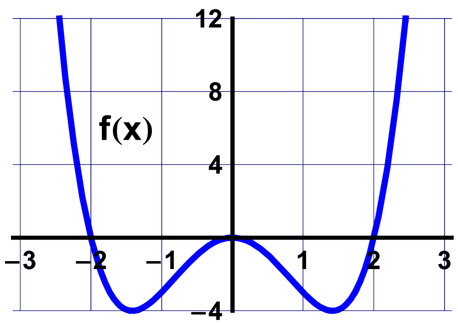
1. Use the graph of the function f to estimate where (a) f' and (b) f'' are zero, positive, and negative.



(a) f' is positive on $(1.5, \infty)$ and on $(-1.5, 0)$, negative on $(-\infty, -1.5)$ and $(0, 1.5)$, and
$f' = 0$ at $x = -1.5$, $x = 0$, and $x = 1.5$

(b) f'' is positive on $(-\infty, -0.8)$ and $(0.8, \infty)$, negative on $(-0.8, 0.8)$, and $f'' = 0$ at $x = -0.8$, and $x = 0.8$

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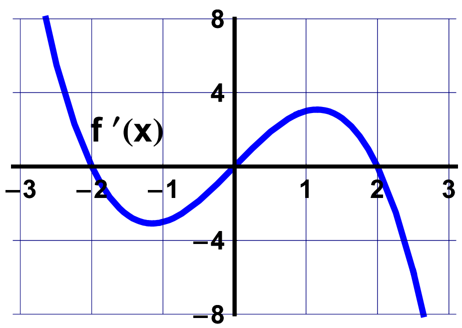
2. Use the graph of f' to estimate where the function f is (a) increasing or (b) decreasing, and (c) estimate where f has local extreme values.



(a) f is increasing on $(-\infty, -2]$ and $[0, 2]$, (b) f is decreasing on $[-2, 0]$ and $[2, \infty)$, and
(c) f has a local maximum at $x = -2$, and $x = 2$, and a local minimum at $x = 0$

For the following problems, use analytic methods to find the intervals on which the function is
(a) increasing, (b) decreasing, (c) concave up, (d) concave down, then find any (e) local extreme values, and
(f) inflection points

9. $y = 2x^4 - 4x^2 + 1$

$y' = 8x^3 - 8x = 8x(x+1)(x-1) \rightarrow c = -1, 0, 1$

$y'' = 24x^2 - 8 = 8(3x^2 - 1) \rightarrow k = \frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$

(a) y is increasing on the intervals $[-1, 0]$ and $[1, \infty)$

(b) y is decreasing on the intervals $(-\infty, -1]$ and $[0, 1]$

(c) y is concave up on the intervals $\left(-\infty, \frac{-\sqrt{3}}{3}\right)$ and $\left(\frac{\sqrt{3}}{3}, \infty\right)$

(d) y is concave down on the interval $\left(\frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

(e) y has a local minimum at $x = -1$, $x = 1$, and
y has a local maximum at $x = 0$, and

(f) y has points of inflection at $x = \frac{-\sqrt{3}}{3}$, $x = \frac{\sqrt{3}}{3}$

23. $y = \tan^{-1} x$

$y' = \frac{1}{1 + x^2} \rightarrow$ no critical values

$y'' = \frac{-2x}{(1 + x^2)^2} \rightarrow k = 0$

(a) y is increasing on the interval $(-\infty, \infty)$

(b) none

(c) y is concave up on the interval $(-\infty, 0)$

(d) y is concave down on the interval $(0, \infty)$

(e) no extreme values

(f) point of inflection at the point $(0, 0)$

25. $y = x^{\frac{1}{3}}(x - 4)$

$y' = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{\frac{-2}{3}} = \frac{4x - 4}{3x^{\frac{2}{3}}} \rightarrow c = 0, 1$

$y'' = \frac{4}{9}x^{\frac{-2}{3}} + \frac{8}{9}x^{\frac{-5}{3}} = \frac{4x + 8}{x^{\frac{5}{3}}} \rightarrow k = -2, 0$

(a) y is increasing on the interval $[1, \infty]$

(b) y is decreasing on the intervals $(-\infty, 0]$, $[0, 1]$

(c) y is concave up on the intervals $(-\infty, -2)$, $(0, \infty)$

(d) y is concave down on the interval $(-2, 0)$

(e) y has a local minimum at $x = 1$, y has no local maximum

(f) y has a point of inflection at $x = -2$

(no PI at $x = 0$, because $f'(0)$ does not exist)

28. $y = \frac{x}{x^2 + 1}$

$y' = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} \rightarrow c = -1, 1$

$y'' = \frac{(x^2 + 1)^2(-2x) - (1 - x^2)(2(x^2 + 1)(2x))}{(x^2 + 1)^4}$ so

$y'' = \frac{(x^2 + 1)(-2x) - (1 - x^2)(2(2x))}{(x^2 + 1)^3}$ so

$y'' = \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2 + 1)^3} = \frac{2x^3 - 6x}{(x^2 + 1)^3}$ so

$y'' = \frac{2x(x - \sqrt{3})(x + \sqrt{3})}{(x^2 + 1)^3} \rightarrow k = -\sqrt{3}, 0, \sqrt{3}$

(a) y is increasing on the interval $[-1, 1]$

(b) y is decreasing on the intervals $(-\infty, -1]$, $[1, \infty)$

(c) y is concave up on the intervals $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$

(d) y is concave down on the intervals $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$

(e) y has a local min at $x = -1$, and a local max at $x = 1$

(f) y has PI's at $x = -\sqrt{3}$, $x = 0$, and $x = \sqrt{3}$