4.4 Modeling and Optimization

How Calculus Saved My Life

distance = rate x time, so $t = \frac{d}{r}$ and $T(x) = \frac{20 - x}{5} + \frac{\sqrt{x^2 + 25}}{4}$ so

$T'(x) = \frac{-1}{5} + \frac{1}{4}\left(\frac{1}{2}\right)(x^2 + 25)^{\frac{-1}{2}}(2x) = 0 \rightarrow \frac{1}{5} = \frac{x}{4\sqrt{x^2 + 25}} \rightarrow 16(x^2 + 25) = 25x^2$

$400 = 9x^2$, so $x = \frac{20}{3}$ or $x = 6\frac{2}{3}$ $\quad T\left(6\frac{2}{3}\right) = 4\frac{3}{4}$ hours $\quad$ Now, test the endpoints, so that

$T(20) = \frac{20 - 20}{5} + \frac{\sqrt{20^2 + 25}}{4} \approx 5.154$ and $T(0) = \frac{20 - 0}{5} + \frac{\sqrt{0^2 + 25}}{4} = 5.25$ so

the minimum is $4\frac{3}{4}$ hours, at $x = 6\frac{2}{3}$ miles

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6. A rectangle has its base on the x – axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

$A = 2x(12 - x^2) \rightarrow A = 24x - 2x^3$ so $A'(x) = 24 - 6x^2 = 6(4 - x^2)$ so $c = 2, -2$

and $A'' = -12x$ $\quad A''(2) = -12(2) = -24 < 0$

Therefore, there is a maximum at $x = 2$ and $A(2) = 32$, so the dimensions are 4 x 8.

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10. A 216 meters^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

$xy = 216$ $\quad P = 3x + 2y$ so $P(x) = 3x + 2\left(\frac{216}{x}\right) = 3x + 432x^{-1}$ so $P'(x) = 3 - 432x^{-2}$

so $3 - \frac{432}{x^2} = 0$ or $3 = \frac{432}{x^2} \rightarrow x^2 = \frac{432}{3} = 144$ so $x = 12$ meters and $y = \frac{216}{12} = 18$ meters

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17. You are designing a 1000 cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be $A = 8r^2 + 2\pi rh$. What is the ratio of h to r for the most economical can?

$A = 8r^2 + 2\pi rh$ and $V = \pi r^2 h = 1000 \rightarrow h = \frac{1000}{\pi r^2}$ so $A = 8r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 8r^2 + 2000r^{-1}$

and $A'(r) = 16r - 2000r^{-2} = 0 \rightarrow \frac{2000}{r^2} = 16r \rightarrow \frac{2000}{16} = r^3 \rightarrow r = 5$

Use the first derivative test to see if $r = 5$ is a minimum, and $h = \frac{40}{\pi}$ so $h : r = \frac{8}{\pi}$

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31. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

$h = y + 3$ $\quad V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi x^2(y+3) = \frac{1}{3}\pi(9 - y^2)(y+3) = \frac{\pi}{3}(-y^3 - 3y^2 + 9y + 27)$

so $V'(y) = \frac{\pi}{3}(-3y^2 - 6y + 9) = \pi(y^2 + 2y - 3) = \pi(y + 3)(y - 1)$ so $y = 1$ and $x = 2\sqrt{2}$