4.5 Linearization and Newton's Method

Linearization

If f is differentiable at $x = a$, then $L(x) = f(a) + f'(a)(x - a)$ is the linearization of f at a.

Newton's Method

1. Guess an approximation to the solution of $f(x) = 0$
2. Find successive approximations with $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Differentials

Assume $y = f(x)$ is differentiable. dx is an independent variable, and $dy = f'(x)\,dx$

□	Exact	Estimated
Absolute Change	$\Delta y = \Delta f = f(a + dx) - f(a)$	$dy = df = f'(a)\,dx$
Relative Change	$\frac{\Delta y}{f(a)} = \frac{\Delta f}{f(a)}$	$\frac{dy}{f(a)} = \frac{df}{f(a)}$
Percentage Change	$100\,\frac{\Delta y}{f(a)} = 100\,\frac{\Delta f}{f(a)}$	$100\,\frac{dy}{f(a)} = 100\,\frac{df}{f(a)}$

For problems 1 and 2, find the linearization $L(x)$ of $f(x)$ at $x = a$

1. $f(\theta) = 2\sin\theta + \cos\theta, \quad x = \dfrac{5\pi}{6}$

$f'(\theta) = 2\cos\theta - \sin\theta$ so

$$f'\left(\frac{5\pi}{6}\right) = 2\cos\frac{5\pi}{6} - \sin\frac{5\pi}{6} = 2\left(\frac{-\sqrt{3}}{2}\right) - \frac{1}{2} \quad \text{so}$$

$$f'\left(\frac{5\pi}{6}\right) = -\sqrt{3} - \frac{1}{2}$$

$$\text{and} \quad f\left(\frac{5\pi}{6}\right) = 2\sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} = 2\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{2} \quad \text{so}$$

$f\left(\dfrac{5\pi}{6}\right) = 1 - \dfrac{\sqrt{3}}{2}$ so the linearization of

$f(\theta) = 2\sin\theta + \cos\theta$ at $x = \dfrac{5\pi}{6}$

is $$\boxed{y = \left(1 - \frac{\sqrt{3}}{2}\right) + \left(-\sqrt{3} - \frac{1}{2}\right)\left(x - \frac{5\pi}{6}\right)}$$

2. $f(x) = \sqrt{1 + 2x}, \quad x = 2$

$$f'(x) = \frac{1}{2}(1 + 2x)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{1 + 2x}}$$

so $f'(2) = \dfrac{\sqrt{5}}{5}$ $\quad f(2) = \sqrt{5}$

and $y - \sqrt{5} = \dfrac{\sqrt{5}}{5}(x - 2)$

so $$\boxed{y = \sqrt{5} + \frac{\sqrt{5}}{5}(x - 2)}$$

3. Choose a linearization at a nearby value, so that the function can be approximated at $x = a$ if

$f(x) = \dfrac{2x}{x + 2}, \quad a = 2.1$

choose $a = 2$, and $$f'(x) = \frac{(x + 2)(2) - (2x)(1)}{(x + 2)^2} = \frac{4}{(x + 2)^2}$$

so $f'(2) = \dfrac{1}{4}$ and $f(2) = 1$ so the linearization of $f(x) = \dfrac{2x}{x + 2}$ at $a = 2$ is

$$\boxed{y = 1 + \frac{1}{4}(x - 2)} \quad \text{and} \quad y(2.1) = 1 + \frac{1}{4}(2.1 - 2) = \boxed{\frac{41}{40}}$$

Compare this to the true function value at $x = 2.1 \;\rightarrow\; f(2.1) = \dfrac{4.2}{4.1} = \dfrac{42}{41}$

For problems 4 and 5, use Newton′s Method to estimate the solution to the equation on the given interval.

4. $x^3 - 36x - 84 = 0$ $[6, 8]$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ and $f'(x) = 3x^2 - 36$

choosing $\boxed{x_1 = 7,}$ we have $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$x_2 = 7 - \dfrac{343 - 252 - 84}{147 - 36} = 7 - \dfrac{7}{111} = \dfrac{770}{111}$ so

$\boxed{x_2 \approx 6.936937}$ and $x_3 = 6.936937 - \dfrac{f(6.936937)}{f'(6.936937)}$

so $\boxed{x_3 \approx 6.936}$

5. $x^5 - 2x^2 + 4 = 0$ $[-2, -1]$

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ and $f'(x) = 5x^4 - 4x$

choosing $\boxed{x_1 = -2,}$ $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$x_2 = -2 - \dfrac{-32 - 8 + 4}{80 + 8} = -2 + \dfrac{9}{22} \approx \boxed{-1.59091}$

$x_3 = -1.59091 - \dfrac{f(-1.59091)}{f'(-1.59091)}$

so $\boxed{x_3 \approx -1.298}$

For problems 6 and 7, find (a) Δy, and (b) dy, and (c) $\Delta y - dy$.

6. $f(x) = x^2 + 3x$, $a = 1$, $\Delta x = 0.1$

(a) $\Delta y = f(1.1) - f(1) = 1.1^2 + 3(1.1) - (1^2 + 3(1))$

so $\boxed{\Delta y = 0.51}$

(b) $dy = f'(x)\,dx = (2x + 3)\Delta x = (2(1) + 3)(0.1)$

so $\boxed{dy = 0.5,}$ and

(c) $\Delta y - dy = 0.51 - 0.5 = \boxed{0.01}$

7. $f(x) = x(\ln(x))$, $a = e$, $\Delta x = -0.1$

(a) $\Delta y = f(e - 0.1) - f(e) = (e - 0.1)\ln(e - 0.1) - e\ln e$

so $\Delta y = (e - 0.1)\ln(e - 0.1) - e \approx \boxed{-0.198}$

(b) $dy = f'(x)\,dx = \left(\ln(x) + x\left(\dfrac{1}{x}\right)\right)\Delta x = ((\ln x) + 1)\Delta x$

so $dy = ((\ln e) + 1)(-0.1) = \boxed{-0.2}$

(c) $\Delta y - dy \approx -0.198 - (-0.2) \approx \boxed{0.002}$

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8. Write a differential formula that estimates the change in volume, $V = \dfrac{4}{3}\pi r^3$, when the radius changes from a to $a + dr$

$\Delta V = \dfrac{4}{3}\pi(a + \Delta r)^3 - \dfrac{4}{3}\pi a^3$, but we are looking for $dV = V'(r)\,dr = 4\pi r^2\,dr \rightarrow \boxed{4\pi a^2\,dr}$

and $dV \approx \Delta V$ as long as dr is small

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9. About how accurately should you measure the side of a square to be sure of calculating the area to within 2% of its true value?

$A = s^2$ and $dA = 2s(\Delta s)$ and we know that $\left|100\dfrac{dy}{y}\right| \le 2$ or $\left|100\dfrac{dA}{A}\right| \le 2$

so $\left|100\dfrac{2s\,ds}{s^2}\right| \le 2 \rightarrow \left|100\dfrac{ds}{s}\right| \le 1 \rightarrow \left|\dfrac{ds}{s}\right| \le \dfrac{1}{100}$ so measure the side to within $\boxed{1\%}$