4.6 Related Rates

How Calculus Saved My Job (and my future)

$$x^2 + y^2 = z^2 \quad \to \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \quad \to \quad x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt} \quad \text{so}$$

$$(400\text{ ft})\frac{dx}{dt} + (300\text{ ft})\left(44\frac{\text{ft}}{\text{sec}}\right) = (500\text{ ft})\left(102\frac{2}{3}\frac{\text{ft}}{\text{sec}}\right) \qquad \boxed{\frac{dx}{dt} = 95\frac{1}{3}\frac{\text{ft}}{\text{sec}} = 65\text{ mph}}$$

3. The volume of a right circular cylinder is $V = \pi r^2 h$, so find

(a) $\frac{dV}{dt}$ if r is constant, and (b) $\frac{dV}{dt}$ if h is constant, and (c) $\frac{dV}{dt}$ if neither r nor h are constant

(a) $\boxed{\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}}$ (b) $\boxed{\frac{dV}{dt} = \pi h\left(2r\frac{dr}{dt}\right)}$ (c) $\boxed{\frac{dV}{dt} = \pi\left(2r\frac{dr}{dt}\right)h + \pi r^2\frac{dh}{dt}}$

12. A trough is 15 feet long and 4 feet across the top. Its ends are isosceles triangles with height 3 feet. Water runs into the trough at the rate of $2.5\ \frac{\text{feet}^3}{\text{minute}}$. How fast is the water level rising when it is 2 feet deep?

$$\frac{dV}{dt} = 2.5\frac{\text{ft}^3}{\text{min}} \quad \text{and} \quad h = 2\text{ ft}, \text{ trying to find } \frac{dh}{dt}$$

$$V = \frac{1}{2}bh(15) \quad \text{because there are similar triangles,} \quad \frac{b}{h} = \frac{4}{3} \quad \to \quad b = \frac{4}{3}h$$

$$V = \frac{15}{2}\left(\frac{4}{3}h\right)h \quad \to \quad V = 10h^2 \quad \text{so} \quad \frac{dV}{dt} = 20h\frac{dh}{dt}$$

$$\text{substitute } h = 2\text{ ft and } \frac{dV}{dt} = 2.5\frac{\text{ft}^3}{\text{min}}, \quad 2.5 = 20(2)\frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{25}{400} = \boxed{\frac{1}{16}\frac{\text{ft}}{\text{min}}}$$

21. A dinghy (small boat) is pulled toward a dock by a rope from the bow through a ring on the dock 6 feet above the bow. The rope is hauled in at the rate of $2\ \frac{\text{feet}}{\text{second}}$. (a) How fast is the boat approaching the dock when 10 feet of rope are out? (b) At what rate is angle θ changing at that moment?

$$x^2 + y^2 = z^2 \quad \to \quad x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}, \quad \text{and } x = 8,\ y = 6,\ z = 10$$

$$\frac{dz}{dt} = 2\frac{\text{ft}}{\text{sec}}, \quad \frac{dy}{dt} = 0\frac{\text{ft}}{\text{sec}}, \quad \frac{dx}{dt} = ? \quad \text{so} \quad 8\frac{dx}{dt} + 6(0) = 10(2)$$

Therefore, $\boxed{\frac{dx}{dt} = 2.5\frac{\text{ft}}{\text{sec}}}$ (b) $\cos(\theta) = \frac{6}{z} = 6z^{-1} \quad \to \quad -\sin(\theta)\left(\frac{d\theta}{dt}\right) = \frac{-6}{z^2}\left(\frac{dz}{dt}\right)$

$$-\sin(\theta)\left(\frac{d\theta}{dt}\right) = \frac{-6}{10^2}(2) \quad \to \quad \sin(\theta)\left(\frac{d\theta}{dt}\right) = \frac{3}{25} \quad \to \quad \frac{d\theta}{dt} = \frac{3}{25}\csc\theta = \frac{3}{25}\left(\frac{10}{8}\right)$$

so $\boxed{\frac{d\theta}{dt} = \frac{3}{20}\frac{\text{radians}}{\text{sec}}}$

24. Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of $10\ \frac{\text{inches}^3}{\text{minute}}$. (a) How fast is the level in the pot rising when the coffee in the cone is 5 inches deep? (b) How fast is the level in the cone falling at that moment?

(a) $\frac{dV}{dt} = 10\,\frac{m^3}{min}$ $\quad V_{cylinder} = \pi r^2 h$, and $r = 3$ → $V = 9\pi h$ → $\frac{dV}{dt} = 9\pi\,\frac{dh}{dt}$ so $\boxed{\frac{dh}{dt} = \frac{10}{9\pi}\,\frac{in}{min}}$

(b) $V_{cone} = \frac{1}{3}\pi r^2 h$, and through similar triangle ratios, $\frac{r}{h} = \frac{3}{6}$ → $r = \frac{1}{2}h$

$V = \frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12}h^3$ → $\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt}$ → $\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$

so $10 = \frac{\pi}{4}25\frac{dh}{dt}$ → $\boxed{\frac{dh}{dt} = \frac{10(4)}{25\pi} = \frac{8}{5\pi}\,\frac{in}{min}}$

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30. A light shines from the top of a pole 50 feet high. A ball is dropped from the same height from a point 30 feet away from the light. How fast is the shadow of the ball moving along the ground 0.5 seconds later (assume that the ball falls a distance $s = 16t^2$ feet in t seconds) ?

$\frac{ds}{dt} = 32t$ → $s\left(\frac{1}{2}\right) = 4$ ft and $\frac{ds}{dt}$ at $t = \frac{1}{2}$ → $16\,\frac{ft}{sec}$

Using similar triangle ratios, $\frac{50}{x} = \frac{50-s}{x-30}$ → $50x - sx = 50x - 1500$

so $sx = 1500$ → $x = 1500s^{-1}$ and $\frac{dx}{dt} = \frac{-1500}{s^2}\frac{ds}{dt}$ so

$\frac{ds}{dt} = \frac{-1500}{16}(16)$ so $\boxed{\frac{ds}{dt} = -1500\,\frac{ft}{sec}}$