5.3 Definite Integrals and Antiderivatives

Rules for Definite Integrals

1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

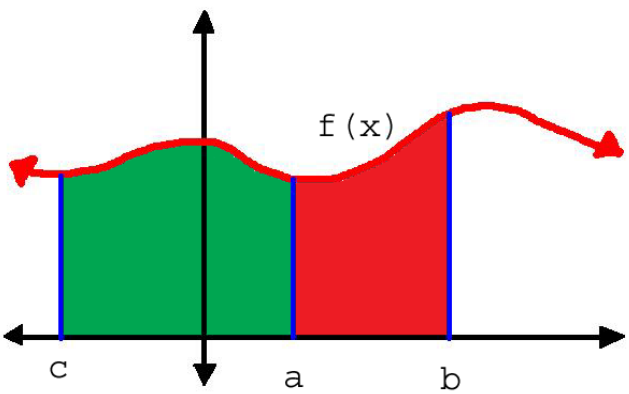
3. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ for any real k

4. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. $(\text{min of } f)(b-a) \le \int_a^b f(x)\,dx \le (\text{max of } f)(b-a)$

7. If $f(x) \ge g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ If $f(x) \ge 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \ge 0$



$\int_c^a f(x)\,dx + \int_a^b f(x)\,dx = \int_c^b f(x)\,dx$ so

$\int_a^b f(x)\,dx - \int_c^b f(x)\,dx = -\int_c^a f(x)\,dx$ so $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Assume that $\int_{-2}^1 h(x)\,dx = 3$, $\int_5^1 h(x)\,dx = 4$, and $\int_1^5 g(x)\,dx = -6$, and find . . .

1. $\int_1^5 (h(x) - g(x))\,dx$

$\int_1^5 h(x)\,dx - \int_1^5 g(x)\,dx = -4 - (-6) = \boxed{2}$

2. $\int_5^1 (-3\,g(x))\,dx$

$3\int_1^5 g(x)\,dx = 3(-6) = \boxed{-18}$

3. $\int_1^5 (2g(x) - 4h(x))\,dx$

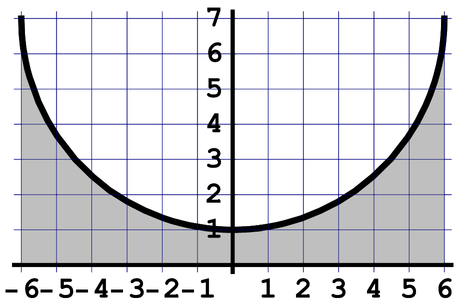
$2\int_1^5 g(x)\,dx - 4\int_1^5 h(x)\,dx = 2(-6) - 4(-4) = \boxed{4}$

4. $\int_5^{-2} \left(\frac{h(x)}{3}\right) dx$

$\frac{-1}{3}\int_{-2}^5 h(x)\,dx = \frac{-1}{3}(3 + (-4)) = \boxed{\frac{1}{3}}$

Don't forget your geometry skills...

5. $\int_{-6}^{6} \left(7 - \sqrt{36 - x^2}\right) dx$



$= 12(7) - \frac{1}{2}(\pi 36) = \boxed{84 - 18\pi}$

So now, let's evaluate definite integrals in a new way...

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $\int_a^x f(t)\,dt = F(x) - F(a)$ where $F'(x) = f(x)$

(we'll make use of this today, and prove it the next day in class)

That's right, to evaluate a definite integral, we can find an antiderivative!

6. $\int_1^3 e^{2x}\,dx$

$= \left[\frac{e^{2z}}{2}\right]_1^3$

$= \boxed{\frac{1}{2}(e^6 - e^2)}$

7. $\int_{-2}^{1} \frac{2}{x+3}\,dx$

$= \left[2\ln(x+3)\right]_{-2}^{1}$

$= \boxed{2(\ln 4)}$

8. $\int_1^2 \frac{4}{x\sqrt{x^2 - 1}}\,dx$

$= 4\left[\sec^{-1} x\right]_1^2$

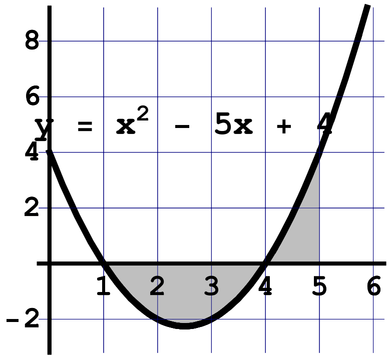
$= 4\left(\frac{\pi}{3} - 0\right) = \boxed{\frac{4\pi}{3}}$

9. $\int_{\frac{-\pi}{6}}^{\frac{\pi}{2}} (-3\sin x)\,dx$

$= 3\left[\cos x\right]_{\frac{-\pi}{6}}^{\frac{\pi}{2}}$

$= 3\left(0 - \frac{\sqrt{3}}{2}\right) = \boxed{\frac{-3\sqrt{3}}{2}}$

For the following problems, find the total shaded area.



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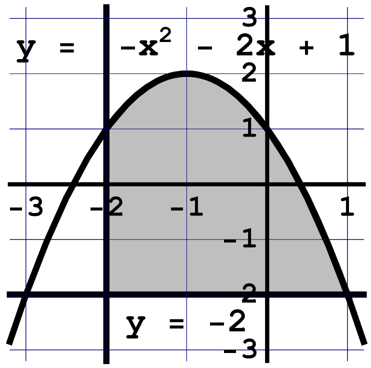

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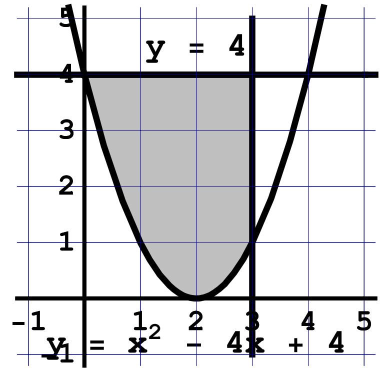


12.

10.

$$= -\int_1^4 (x^2 - 5x + 4)\,dx + \int_4^5 (x^2 - 5x + 4)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{5}{2}x^2 + 4x\right]_4^1 + \left[\frac{x^3}{3} - \frac{5}{2}x^2 + 4x\right]_4^5$$

$$= \left(\frac{1}{3} - \frac{5}{2} + 4\right) - \left(\frac{64}{3} - 40 + 16\right) + \left(\frac{125}{3} - \frac{125}{2} + 20\right) - \left(\frac{64}{3} - 40 + 16\right)$$

$$= \frac{-2}{3} - 65 + 80 - 32 + 24 = \boxed{\frac{19}{3}}$$

11.

$$= \int_{-2}^1 ((-x^2 - 2x + 1) - (-2))\,dx$$

$$= \int_{-2}^1 (-x^2 - 2x + 3)\,dx$$

$$= \left[\frac{-x^3}{3} - x^2 + 3x\right]_{-2}^1$$

$$= \left(\frac{-1}{3} - 1 + 3\right) - \left(\frac{8}{3} - 4 - 6\right)$$

$$= -3 + 2 + 10 = \boxed{9}$$

12.

$$= \int_0^3 (4 - (x^2 - 4x + 4))\,dx$$

$$= \int_0^3 (4x - x^2)\,dx$$

$$= \left[2x^2 - \frac{1}{3}x^3\right]_0^3$$

$$= 18 - 9 = \boxed{9}$$

Average Value: If $f(x)$ is integrable on $[a, b]$, then $f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Mean Value Theorem for Definite Integrals

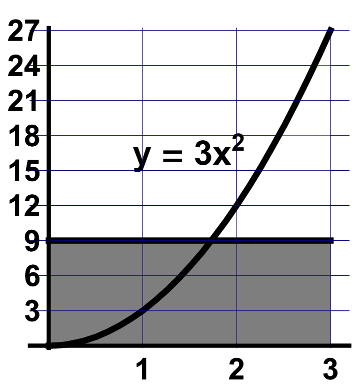
If $f(x)$ is continuous on $[a, b]$, then there exists a value c on $[a, b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

For the following problems, find the average value of the function over the given interval. Also, find the points on the interval where this average value occurs (the c values).

13. $f(x) = 3x^2$ on $[0, 3]$

$$f_{av} = \frac{1}{3-0}\int_0^3 3x^2\,dx = \frac{1}{3}\left[x^3\right]_0^3 = \frac{1}{3}(27 - 0) = \boxed{9}$$

and so $f(c) = 3c^2 = 9$ so $\boxed{c = \sqrt{3}}$



14. $f(x) = \frac{3}{x^2}$ on $[-4, -1]$

$$f_{av} = \frac{1}{-1-(-4)} \int_{-4}^{-1} \frac{3}{x^2}\, dx$$

$$= \frac{1}{3}\, 3 \int_{-4}^{-1} \frac{1}{x^2}\, dx = -\left[\frac{1}{x}\right]_{-4}^{-1}$$

$$= -\left(-1 - \left(\frac{-1}{4}\right)\right) = \boxed{\frac{3}{4}}$$ and

$f(c) = \frac{3}{4}$ or $\frac{3}{c^2} = \frac{3}{4}$ so

$\boxed{c = -2}$

15. $f(x) = 3\sqrt{x+1}$ on $[-1, 8]$

$$f_{av} = \frac{1}{8-(-1)} \int_{-1}^{8} 3\sqrt{x+1}\, dx$$

$$= \frac{1}{3} \int_{-1}^{8} \sqrt{x+1}\, dx =$$

$$\frac{2}{9}\left[(x+1)^{\frac{3}{2}}\right]_{-1}^{8} = \frac{2}{9}(27 - 0)$$

$= \boxed{6}$ so $f(c) = 6$ or

$3\sqrt{c+1} = 6$ or $\boxed{c = 3}$

16. $f(x) = \begin{cases} 2x + 4 & x < 0 \\ -2x + 4 & x \ge 0 \end{cases}$ on $[-3, 3]$

$$f_{av} = \frac{1}{3-(-3)}\left(\int_{-3}^{0} (2x+4)\, dx + \int_{0}^{3} (-2x+4)\, dx\right)$$

$$= \frac{1}{6}\left(\left[x^2 + 4x\right]_{-3}^{0} + \left[-x^2 + 4x\right]_{0}^{3}\right)$$

$$= \frac{1}{6}(3 + 3) = \boxed{1}$$ so

$f(c) = 1$ so $2c + 4 = 1$ or $-2c + 4 = 1$

so $\boxed{c = \frac{-3}{2}, \frac{3}{2}}$