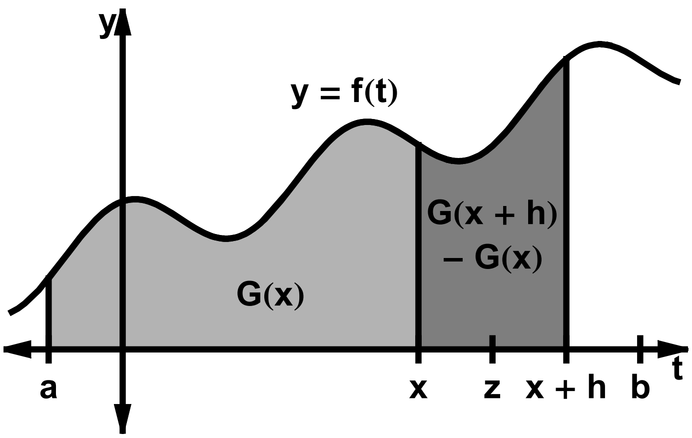
5.4 Fundamental Theorem of Calculus

I. If f is continuous on [a, b], then the function $G(x) = \int_a^x f(t)\,dt$

has a derivative at every point x in [a, b], and $\frac{dG}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

II. If f is continuous at every point of [a, b], and if F is any antiderivative

of f on [a, b], then $\int_a^b f(x)\,dx = F(b) - F(a)$



Sketch of the proof: We want to show that $G'(x) = f(x)$, or that $\lim_{h \to 0} \frac{G(x+h) - G(x)}{h} = f(x)$

I. $G(x) = \int_a^x f(t)\,dt \quad G(x+h) = \int_a^{x+h} f(t)\,dt$ so $G(x+h) - G(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$

so $\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$ and by the Mean Value Theorem,

$\int_x^{x+h} f(t)\,dt = f(z)(x + h - x) = h\,f(z) \rightarrow f(z) = \frac{1}{h}\int_x^{x+h} f(t)\,dt$ so $\frac{G(x+h) - G(x)}{h} = f(z)$ or

$\lim_{h \to 0^+} \frac{G(x+h) - G(x)}{h} = \lim_{h \to 0^+} f(z) = f(x)$ and $\boxed{G'(x) = f(x)}$

II. Assume that F (x) is an antiderivate of f (x) $\quad G(x) = F(x) + c$

$F(x) + c = \int_a^x f(t)\,dt \rightarrow$ plug in x = a $\quad F(a) + c = \int_a^a f(t)\,dt = 0 \quad c = -F(a)$

$F(x) - F(a) = \int_a^x f(t)\,dt \rightarrow$ plug in x = b $\quad \boxed{F(b) - F(a) = \int_a^b f(t)\,dt}$

1. $\displaystyle\int_1^4 (\ln 4)\, 4^x\, dx$

$= \left[\, 4^x \,\right]_1^4$

$= 256 - 4$

$= \boxed{252}$

2. $\displaystyle\int_0^{\frac{\pi}{6}} \frac{2\sin x}{\cos^2 x}\, dx$

$= 2\displaystyle\int_0^{\frac{\pi}{6}} \sec x \tan x \, dx$

$= 2\left[\, \sec x \,\right]_0^{\frac{\pi}{6}}$

$= \boxed{2\left(\frac{2\sqrt{3}}{3} - 1\right)}$

3. $\displaystyle\int_{-2}^{-1} \left(x - \frac{1}{x}\right)^2 dx$

$= \displaystyle\int_{-2}^{-1} \left(x^2 - 2 + \frac{1}{x^2} \right) dx$

$= \left[\, \frac{x^3}{3} - 2x - \frac{1}{x} \,\right]_{-2}^{-1}$

$= \left(\frac{-1}{3} + 2 + 1\right) - \left(\frac{-8}{3} + 4 + \frac{1}{2}\right)$

$= \frac{7}{3} - \frac{3}{2} = \boxed{\frac{5}{6}}$

4. $\displaystyle\int_{-1}^{-2} \frac{2t - 7}{t^3}\, dt$

$= \displaystyle\int_{-1}^{-2} \left(\frac{2}{t^2} - \frac{7}{t^3} \right) dt$

$= \left[\, \frac{-2}{t} + \frac{7}{2t^2} \,\right]_{-1}^{-2}$

$= \left(1 + \frac{7}{8}\right) - \left(2 + \frac{7}{2}\right)$

$= \frac{15}{8} - \frac{11}{2} = \boxed{\frac{-29}{8}}$

5. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (4\sin 2\theta + 6\cos 3\theta)\, d\theta$

$= \left[\, -2\cos 2\theta + 2\sin 3\theta \,\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \left(-2\left(\frac{-1}{2}\right) + 2(0)\right) - \left(-2(0) + 2\left(\frac{\sqrt{2}}{2}\right)\right)$

$= \boxed{1 - \sqrt{2}}$

6. $\displaystyle\int_{-3}^{0} \frac{x^3 + 8}{x + 2}\, dx$

$= \displaystyle\int_{-3}^{0} \frac{(x + 2)(x^2 - 2x + 4)}{(x + 2)}\, dx$

$= \displaystyle\int_{-3}^{0} (x^2 - 2x + 4)\, dx$

$= \left[\, \frac{x^3}{3} - x^2 + 4x \,\right]_{-3}^{0} = 0 - (-9 - 9 - 12) = \boxed{30}$

7. $\displaystyle\int_0^{\pi} \sec x\, dx$ → **Cannot be determined**

$D_x \displaystyle\int_a^x f(t)\, dt = f(x)$ and $D_x \displaystyle\int_a^{g(x)} f(t)\, dt = f(g(x))\, g'(x)$ and $D_x \displaystyle\int_{h(x)}^{g(x)} f(t)\, dt = f(g(x))\, g'(x) - f(h(x))\, h'(x)$

8. $D_x \displaystyle\int_0^{x^3} e^{t^2}\, dt$

$= e^{(x^3)^2}(3x^2)$

$= \boxed{e^{x^6}(3x^2)}$

9. $\displaystyle\frac{d}{dx}\int_{\frac{\pi}{3}}^{x} \cot s\, ds$

$= \boxed{\cot x}$

10. $D_x \displaystyle\int_{x^2}^{\sqrt{x}} (5t - 3)\, dt$

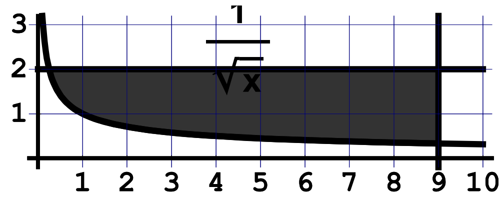
$= \boxed{\left(5\sqrt{x} - 3\right)\frac{1}{2\sqrt{x}} - (5x^2 - 3)\, 2x}$

11. $D_t \displaystyle\int_{\sin^2 t}^{\cos(t^2)} (x^2 - 3x + 2)\, dx$

$= \boxed{\left((\cos(t^2))^2 - 3\cos(t^2) + 2\right)(-\sin(t^2))\, 2t - (\sin^4 t - 3\sin^2 t + 2)(2\sin t \cos t)}$

Find the area of the shaded regions

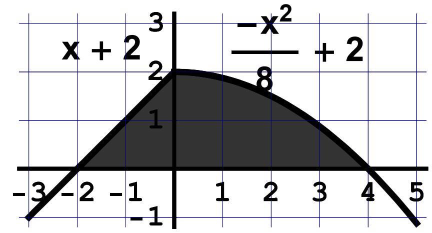
12.



$$\int_{\frac{1}{4}}^{9} \left(2 - x^{\frac{-1}{2}}\right) dx \;=\; \Big[\, 2x - 2\sqrt{x} \,\Big]_{\frac{1}{4}}^{9}$$

$$= (18 - 6) - \left(\frac{1}{2} - 1\right) = \boxed{12\,\frac{1}{2}}$$

13.



$$= \int_{-2}^{0} (x + 2)\, dx + \int_{0}^{4} \left(\frac{-x^2}{8} + 2\right) dx$$

$$= \Big[\, \frac{x^2}{2} + 2x \,\Big]_{-2}^{0} + \Big[\, \frac{-x^3}{24} + 2x \,\Big]_{0}^{4}$$

$$= 0 - (2 - 4) + \left(\frac{-8}{3} + 8\right) - 0 = \frac{22}{3} = \boxed{7\,\frac{1}{3}}$$