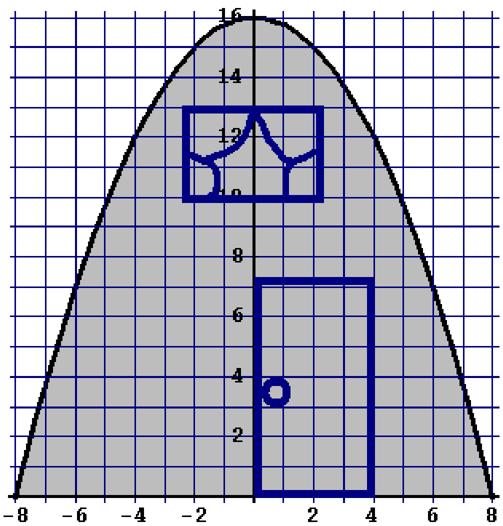
Finding the exact area under a linear, quadratic, or cubic function.

Some expansion formulas: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ $\quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ $\quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

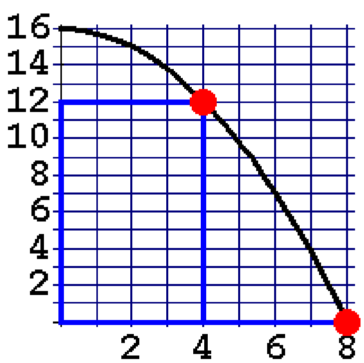
For a continuous function $f(x)$ on the interval $[a, b]$, the "area underneath the curve" can be found by

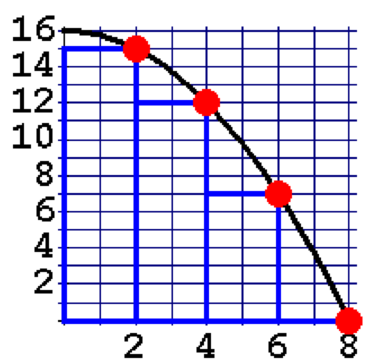
$$\text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n} k\right)\left(\frac{b-a}{n}\right)$$

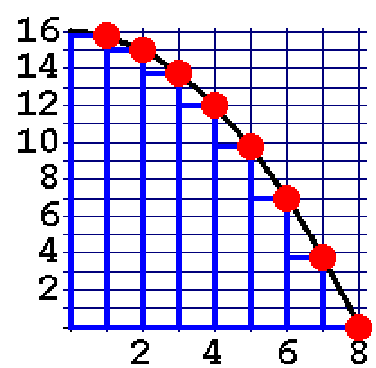


1. How much paint should we use to paint Grandma's house? We know that one quart will cover 4 square feet, so we need to find the area of $\frac{1}{4}$ of the house, or $\frac{1}{2}$ of the front.

Let's try some approximations first . . .







$A_{approx1} = \frac{8-0}{2}(12+0) = \boxed{48}$

$A_{approx2} = \frac{8-0}{4}(f(2) + f(4) + f(6) + f(8))$
$= 2(15 + 12 + 7 + 0) = \boxed{68}$

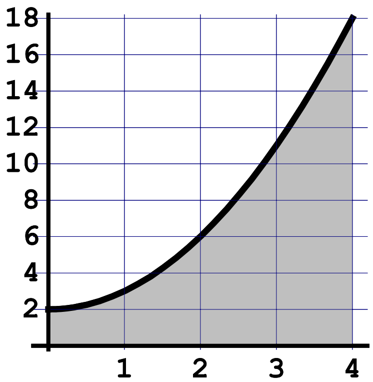
$A_{approx3} = \frac{8-0}{8}(f[1] + f[2] + f[3] + f[4] +$
$f[5] + f[6] + f[7] + f[8]) = \boxed{77}$

Now, let's find the exact area using a limiting process.

$$\text{Area} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{(8-0)}{n}\left(16 - \frac{\left(0 + \frac{8}{n} k\right)^2}{4}\right) = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{128}{n} - \frac{128 k^2}{n^3}\right)$$

$$128 - \lim_{n \to \infty} \frac{128}{n^3} \frac{n(n+1)(2n+1)}{6} = 128 - \frac{128}{3} = \boxed{85\frac{1}{3} \text{ quarts!}}$$

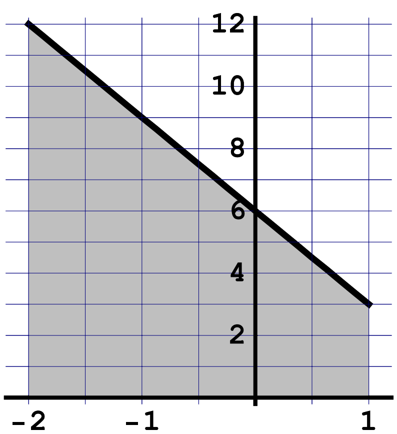
2. Find the area under the curve $y = x^2 + 2$ on the interval $[0, 4]$



$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{4-0}{n}\left(\left[0+\frac{4}{n}k\right]^2+2\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{4}{n}\left(\frac{16k^2}{n^2}+2\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{64k^2}{n^3} + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{8}{n} =$$

$$\lim_{n\to\infty}\frac{64}{n^3}\,\frac{n(n+1)(2n+1)}{6} + 8 = \frac{64}{3} + 8 = \frac{88}{3} = \boxed{29\,\frac{1}{3}}$$

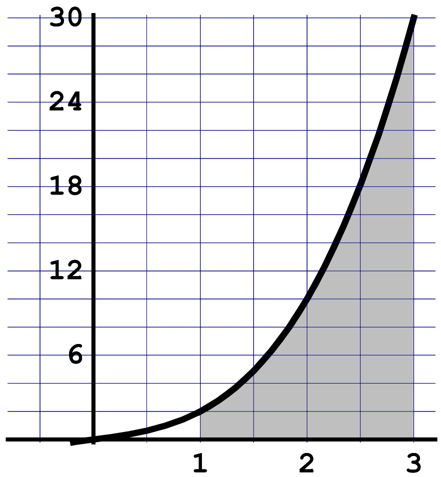
3. Find the area under the curve $y = 6 - 3x$ on the interval $[-2, 1]$



$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1-(-2)}{n}\left(6-3\left[-2+\frac{3}{n}k\right]\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{3}{n}\left(12-\frac{9k}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{36}{n}-\frac{27k}{n^2}\right) = 36 - \lim_{n\to\infty}\frac{27}{n^2}\,\frac{n(n+1)}{2} = 36 - 13.5 = \boxed{22\,\frac{1}{2}}$$

4. Find the area under the curve $y = x^3 + x$ on the interval $[1, 3]$



$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\left(\left[1+\frac{2}{n}k\right]^{3}+\left[1+\frac{2}{n}k\right]\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\left(1+6\frac{k}{n}+12\frac{k^2}{n^2}+8\frac{k^3}{n^3}+1+\frac{2}{n}k\right) =$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{4}{n}+\frac{16k}{n^2}+\frac{24k^2}{n^3}+\frac{16k^3}{n^4}\right) = 4+\lim_{n\to\infty}\frac{16}{n^2}\frac{n(n+1)}{2}+\lim_{n\to\infty}\frac{24}{n^3}\frac{n(n+1)(2n+1)}{6}+\lim_{n\to\infty}\frac{16}{n^4}\left(\frac{n(n+1)}{2}\right)^2 =$$

$$4+8+8+4 = \boxed{24}$$