From Sums to Integrals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n}k\right)\left(\frac{b-a}{n}\right)$$

Remember, that in this formula, $x = a + \frac{b-a}{n}k$, and $\Delta x = dx = \frac{b-a}{n}$ Look for the x first!

To be more specific, you are looking for $a + \frac{b-a}{n}k$, so something like $2 + \frac{3k}{n}$ or $\frac{2k}{n}$ or $5 + \frac{k}{n}$ or . . .

1. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^4 \left(\frac{1}{n}\right)$

$a + \frac{b-a}{n}k = \frac{k}{n}$ so

$a = 0$ and $b = 1$ and

$f(x) = x^4$ so $\int_0^1 x^4\,dx$

$= \frac{1}{5}\left[x^5\right]_0^1 = \boxed{\frac{1}{5}}$

2. $\lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{k=1}^{n} \frac{1}{1 + \left(\frac{k}{n}\right)^2}$

$a + \frac{b-a}{n}k = \frac{k}{n}$ so

$a = 0$ and $b = 1$ and

$f(x) = \frac{1}{1+x^2}$ so $\int_0^1 \frac{1}{1+x^2}\,dx$

$= \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$

3. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(\left(2 + \frac{k}{n}\right)^2 + 3\left(2 + \frac{k}{n}\right)\right)$

$a + \frac{b-a}{n}k = 2 + \frac{k}{n}$ so

$a = 2$ and $b = 3$ and

$2\int_2^3 (x^2 + 3x)\,dx = 2\left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_2^3$

$= 2\left(\left(9 + \frac{27}{2}\right) - \left(\frac{8}{3} + 6\right)\right)$

$= 6 + 27 - \frac{16}{3} = \boxed{27\frac{2}{3}}$

4. $\lim_{n\to\infty} \left(\frac{6}{n}\right) \sum_{k=1}^{n} \left(\frac{n}{3k+n}\right)^2$

$= \lim_{n\to\infty} \left(\frac{6}{n}\right) \sum_{k=1}^{n} \left(\frac{n+3k}{n}\right)^{-2}$

$a + \frac{b-a}{n}k = 1 + \frac{3k}{n}$ so

$a = 1$ and $b = 4$ and

$f(x) = x^{-2}$ so $2\int_1^4 x^{-2}\,dx$

$= -2\left[\frac{1}{x}\right]_1^4 = -2\left(\frac{1}{4} - 1\right)$

$= \boxed{\frac{3}{2}}$

5. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3\left(1 + \frac{2k}{n}\right)^2 - 6\right)\left(\frac{2}{n}\right)$

$a + \frac{b-a}{n}k = 1 + \frac{2k}{n}$ so

$a = 1$ and $b = 3$ and

$f(x) = 3x^2 - 6$ so $\int_1^3 (3x^2 - 6)\,dx$

$= \left[x^3 - 6x\right]_1^3$

$= ((27 - 18) - (1 - 6))$

$= \boxed{14}$

6. $\lim_{n\to\infty} \left(\frac{3}{n}\right) \sum_{k=1}^{n} \frac{1}{2\left(4 + \frac{2k}{n}\right) - 7}$

$a + \frac{b-a}{n}k = 4 + \frac{2k}{n}$ so

$a = 4$ and $b = 6$ and $\frac{3}{n} = \left(\frac{3}{2}\right)\frac{2}{n}$

$f(x) = \frac{1}{2x-7}$ so $\frac{3}{2}\int_4^6 \frac{1}{2x-7}\,dx$

$= \frac{3}{4}\left[\ln(2x-7)\right]_4^6$

$= \frac{3}{4}(\ln 5 - \ln 1)$

$= \boxed{\frac{3}{4}\ln 5}$

7. $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4}{n}\right)\left(2 + 3\left(-1 + \frac{4k}{n}\right) - \left(-1 + \frac{4k}{n}\right)^2\right)$

$a + \frac{b-a}{n}k = -1 + \frac{4k}{n}$ so $a = -1$ and $b = 3$

and $f(x) = 2 + 3x - x^2$ so $\int_{-1}^{3} (2 + 3x - x^2)\,dx$

$= \left[2x + \frac{3}{2}x^2 - \frac{x^3}{3} \right]_{-1}^{3} = \left(6 + \frac{27}{2} - 9\right) - \left(-2 + \frac{3}{2} + \frac{1}{3}\right)$

$= 12 - 3 + 2 - \frac{1}{3} = \boxed{10\frac{2}{3}}$

8. $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \cos\left(\frac{\pi}{3} + \frac{\pi k}{2n}\right)\left(\frac{2}{n}\right)$

$a + \frac{b-a}{n}k = \frac{\pi}{3} + \frac{\frac{\pi}{2}k}{n}$ so $a = \frac{\pi}{3}$ and $b = \frac{5\pi}{6}$

and $\frac{2}{n} = \left(\frac{4}{\pi}\right)\frac{\frac{\pi}{2}}{n}$ and $f(x) = \cos x$ so

$\frac{4}{\pi}\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \cos x\,dx = \frac{4}{\pi}\left[\sin x \right]_{\frac{\pi}{3}}^{\frac{5\pi}{6}}$

$= \frac{4}{\pi}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \boxed{\frac{2 - 2\sqrt{3}}{\pi}}$