6.2 Integration by Substitution

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du \qquad \text{where } u = g(x) \text{ and } du = g'(x)\, dx$$

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du \qquad \text{where } u = g(x) \text{ and } du = g'(x)\, dx$$

The trick is to use up the whole integrand (and dx) with some combination of a function and its derivative. The du (or the derivative) must exist as a multiple. The u (or the function) can be nested as part of a composition, the denominator, or . . .

Evaluate the integral.

1. $\displaystyle\int \frac{5x}{\sqrt{x^2+3}}\, dx$

$u = x^2 + 3 \qquad du = 2x\, dx$

$\rightarrow \frac{5}{2}\, du = 5x\, dx$

$\displaystyle\frac{5}{2}\int u^{\frac{-1}{2}}\, du = \frac{5}{2}\left(2\sqrt{u}\right) + C$

$\rightarrow \boxed{5\sqrt{x^2+3} + C}$

2. $\displaystyle\int \frac{\left(1+\sqrt{x}\right)^3}{\sqrt{x}}\, dx$

$u = 1 + \sqrt{x} \qquad du = \dfrac{1}{2\sqrt{x}}\, dx$

$\rightarrow 2\, du = \dfrac{1}{\sqrt{x}}\, dx$

$\displaystyle 2\int u^3\, du \rightarrow \frac{1}{2}u^4 + C$

$\rightarrow \boxed{\frac{1}{2}\left(1+\sqrt{x}\right)^4 + C}$

3. $\displaystyle\int \frac{x^2-1}{(x^3-3x+1)^6}\, dx$

$\rightarrow u = x^3 - 3x + 1$

$du = (3x^2 - 3)\, dx$

$\frac{1}{3}\, du = (x^2 - 1)\, dx$

$\displaystyle\frac{1}{3}\int u^{-6}\, du \rightarrow \frac{-1}{15}u^{-5} + C$

$\rightarrow \boxed{\dfrac{-1}{15(x^3-3x+1)^5} + C}$

4. $\displaystyle\int \frac{3}{\sqrt{4-5t}}\, dt$

$u = 4 - 5t \rightarrow du = -5\, dt$

$\rightarrow \frac{-3}{5}\, du = 3\, dt$

$\displaystyle\frac{-3}{5}\int u^{\frac{-1}{2}}\, du \rightarrow \frac{-6}{5}\sqrt{u} + C$

$\rightarrow \boxed{\frac{-6}{5}\sqrt{4-5t} + C}$

5. $\displaystyle\int \cos^3 5x \sin 5x\, dx$

$\rightarrow u = \cos 5x$

$\rightarrow du = -5\sin 5x\, dx$

$-\frac{1}{5}\, du = \sin 5x\, dx$

$\displaystyle\frac{-1}{5}\int u^3\, du = \frac{-1}{20}u^4 + C$

$= \boxed{\frac{-1}{20}\cos^4 5x + C}$

6. $\displaystyle\int (s^2+1)^2\, ds$

$u = s^2 + 1 \qquad du = 2s\, ds$

$\frac{1}{2}\, du = s\, ds \rightarrow \displaystyle\int (s^4 + 2s^2 + 1)\, ds$

$\rightarrow \boxed{\frac{1}{5}s^5 + \frac{2}{3}s^3 + s + C}$

7. $\displaystyle\int \frac{1}{x(\log_4 x)^2}\, dx$

$\rightarrow u = \log_4 x \qquad du = \dfrac{1}{x\ln 4}\, dx$

$\rightarrow \ln 4\, du = \frac{1}{x}\, dx$

$\displaystyle\ln 4\int u^{-2}\, du \rightarrow \frac{-\ln 4}{u} + C$

$\rightarrow \boxed{\dfrac{-\ln 4}{\log_4 x} + C}$

8. $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\, dx \qquad u = \frac{x}{3}$

$\rightarrow \displaystyle\int \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\, dx$

$\rightarrow u = \frac{x}{3} \qquad du = \frac{1}{3}\, dx$

$\displaystyle\int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1} u + C$

$= \boxed{\sin^{-1}\left(\frac{x}{3}\right) + C}$

9. $\displaystyle\int \frac{\sin 4x}{\cos 2x}\, dx$

$= \displaystyle\int \frac{2\sin 2x\cos 2x}{\cos 2x}\, dx$

$\rightarrow \displaystyle\int 2\sin 2x\, dx$

$= \boxed{-\cos 2x + C}$

10. $\displaystyle\int \frac{\sin y}{1 + \cos^2 y}\, dy$

$u = \cos y \qquad -du = \sin y\, dy$

$\displaystyle -\int \frac{1}{1+u^2}\, du = -\tan^{-1} u + C$

$= \boxed{-\tan^{-1}(\cos y) + C}$

11. $\displaystyle\int_{-\pi/4}^{\pi/3} \tan x\, dx$

$\displaystyle = \int_{-\pi/4}^{\pi/3} \frac{\sin x}{\cos x}\, dx \qquad u = \cos x$

$-du = \sin x\, dx$

$\displaystyle -\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \frac{1}{u}\, du = -\left[\ln u\right]_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}$

$\displaystyle = \ln 2 + \ln \frac{\sqrt{2}}{2}$

$= \ln 2 + \ln\sqrt{2} - \ln 2 = \boxed{\frac{1}{2}\ln 2}$

12. $\displaystyle\int_{\pi/2}^{5\pi/6} \sqrt{\csc^5\theta}\, \cot\theta\, d\theta$

$\displaystyle = \int_{\pi/2}^{5\pi/6} \sqrt{\csc^3\theta}\, \csc\theta\cot\theta\, d\theta$

$u = \csc\theta \qquad -du = \csc\theta\cot\theta\, d\theta$

$\displaystyle -\int_1^2 u^{\frac{3}{2}}\, du = -\left[\frac{2}{5}u^{\frac{5}{2}}\right]_1^2$

$\displaystyle = \frac{-2}{5}\left(4\sqrt{2} - 1\right)$

$\displaystyle = \boxed{\frac{2 - 8\sqrt{2}}{5}}$

13. $\displaystyle\int_{-\pi/6}^{\pi/6} (x + \sin 5x)\, dx$

odd function from $-a$ to a

so $= \boxed{0}$

14. $\displaystyle\int_0^1 \frac{1}{(3 - 2x)^2}\, dx$

$u = 3 - 2x \qquad du = -2\, dx$

$\displaystyle -\frac{1}{2}du = dx \qquad \frac{-1}{2}\int_3^1 u^{-2}\, du$

$\displaystyle = \frac{1}{2}\left[\frac{1}{u}\right]_3^1 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$

$\displaystyle = \boxed{\frac{1}{3}}$

15. $\displaystyle\int_{-1}^{2} 4^x\sqrt{1 + 4^x}\, dx$

$u = 1 + 4^x \qquad du = \ln 4\ 4^x\, dx$

$\displaystyle \frac{1}{\ln 4}du = 4^x\, dx \rightarrow \frac{1}{\ln 4}\int_{\frac{5}{4}}^{17} u^{\frac{1}{2}}\, du$

$\displaystyle = \frac{2}{3\ln 4}\left[u^{\frac{3}{2}}\right]_{\frac{5}{4}}^{17}$

$\displaystyle = \boxed{\frac{2}{3\ln 4}\left(17\sqrt{17} - \frac{5\sqrt{5}}{8}\right)}$

16. $\displaystyle\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\, dx$

$u = 1 - x^2 \qquad du = -2x\, dx$

$u = \sin^{-1} x \qquad du = \dfrac{1}{\sqrt{1 - x^2}}\, dx$

$\displaystyle\int_0^{\frac{\pi}{6}} u\, du = \frac{1}{2}\left[\, u^2 \,\right]_0^{\frac{\pi}{6}} = \boxed{\frac{\pi^2}{72}}$

17. $\displaystyle\int_1^{3} \frac{e^x + 1}{e^x}\, dx$

$\displaystyle = \int_1^3 (1 + e^{-x})\, dx = \left[\, x - e^{-x} \,\right]_1^3$

$= \left(3 - \dfrac{1}{e^3}\right) - \left(1 - \dfrac{1}{e}\right)$

$= 2 - \dfrac{1}{e^3} + \dfrac{1}{e} = \boxed{\dfrac{2e^3 - 1 + e^2}{e^3}}$

18. $\displaystyle\int_0^{4} x\sqrt{16 - 3x}\, dx$

$u = 16 - 3x \qquad du = -3\, dx$

$-\dfrac{1}{3}\, du = dx \qquad x = \dfrac{1}{3}(16 - u)$

$\displaystyle\frac{-1}{9}\int_{16}^{4} (16 - u)\, u^{\frac{1}{2}}\, du$

$\displaystyle = \frac{-1}{9}\int_{16}^{4} \left(16u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$

$\displaystyle = \frac{-1}{9}\left[\, \frac{32}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} \,\right]_{16}^{4}$

$= \dfrac{-1}{9}\left(\dfrac{256}{3} - \dfrac{64}{5}\right) + \dfrac{1}{9}\left(\dfrac{32}{3}(64) - \dfrac{2}{5}(4^5)\right) = \boxed{\dfrac{3008}{135}}$

19. $\displaystyle\int_{-1}^{1} \frac{x + x^3 + x^5}{1 + x^2 + x^4}\, dx$ odd function from $-a$ to a so $= \boxed{0}$