6.3 Integration by Parts (the Product Rule "backwards")

$$\int u\,dv = uv - \int v\,du$$

If $u = f(x)$ and $v = g(x)$, then $D_x(uv) = u'v + uv'$ or $D_x(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

Now, if we integrate both sides, we get: $\int (D_x(f(x)g(x)))\,dx = \int (f'(x)g(x))\,dx + \int (f(x)g'(x))\,dx$

or $f(x)g(x) = \int g(x)(f'(x)\,dx) + \int f(x)(g'(x)\,dx)$ and, going back to u and v, we have

$uv = \int v\,du + \int u\,dv$ or $\int u\,dv = uv - \int v\,du$

Use u = L ogarithmic
I nverse Trigonometric
P olynomial
E xponential
T rigonometric

The "trick" for using this technique correctly is to choose the dv properly – dv should be the derivative of something times dx. The dv should also take up as much as possible of the integrand.

1. $\int x \tan^{-1} x\,dx$

$u = \tan^{-1} x \qquad dv = x\,dx$

$du = \frac{1}{1 + x^2}\,dx \qquad v = \frac{1}{2}x^2$

$\frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{x^2 + 1}\,dx$

$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2 + 1 - 1}{x^2 + 1}\,dx$

$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{x^2 + 1}\right)dx$

$= \boxed{\frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x - \frac{1}{2}\tan^{-1} x + C}$

2. $\int \sqrt{x}\,\log_2 x\,dx$

$u = \log_2 x \qquad dv = \sqrt{x}\,dx$

$du = \frac{1}{x \ln 2}\,dx \qquad v = \frac{2}{3}x^{\frac{3}{2}}$

$\frac{2}{3}x^{\frac{3}{2}}\log_2 x - \frac{2}{3\ln 2}\int \frac{x^{\frac{3}{2}}}{x}\,dx$

$= \frac{2}{3}x^{\frac{3}{2}}\log_2 x - \frac{2}{3\ln 2}\int x^{\frac{1}{2}}\,dx$

$= \frac{2}{3}x^{\frac{3}{2}}\log_2 x - \frac{2}{3\ln 2}\,\frac{2}{3}x^{\frac{3}{2}} + C$

$= \boxed{\frac{2}{3}x^{\frac{3}{2}}\log_2 x - \frac{4}{9\ln 2}x^{\frac{3}{2}} + C}$

3. $\int (x + 5)\csc^2 4x\,dx$

$u = x + 5 \qquad dv = \csc^2 4x\,dx$

$du = dx \qquad v = \frac{-1}{4}\cot 4x$

$\frac{-1}{4}(x + 5)\cot 4x + \frac{1}{4}\int \cot 4x\,dx$

$= \frac{-1}{4}(x + 5)\cot 4x + \frac{1}{4}\int \frac{\cos 4x}{\sin 4x}\,dx$

$w = \sin 4x$

$\frac{1}{4}\,dw = \cos 4x\,dx$

$\rightarrow \frac{-1}{4}(x + 5)\cot 4x + \frac{1}{16}\int \frac{1}{w}\,dw$

$= \boxed{\frac{-1}{4}(x + 5)\cot 4x + \frac{1}{16}\ln\left|\sin 4x\right| + C}$

4. $\int x^3 \cos 2x \, dx$

Tabular Integration

u	dv
x^3	$\cos 2x$
$3x^2$	$\frac{1}{2}\sin 2x$
$6x$	$-\frac{1}{4}\cos 2x$
6	$-\frac{1}{8}\sin 2x$
0	$\frac{1}{16}\cos 2x$

$$\boxed{\frac{1}{2}x^3\sin 2x + \frac{3}{4}x^2\cos 2x - \frac{3}{4}x\sin 2x - \frac{3}{8}\cos 2x + C}$$

5. $\int x^4 e^{-3x} \, dx$

Tabular Integration

u	dv
x^4	e^{-3x}
$4x^3$	$\frac{-1}{3}e^{-3x}$
$12x^2$	$\frac{1}{9}e^{-3x}$
$24x$	$\frac{-1}{27}e^{-3x}$
24	$\frac{1}{81}e^{-3x}$
0	$\frac{-1}{243}e^{-3x}$

$$\boxed{\frac{-1}{3}x^4 e^{-3x} - \frac{4}{9}x^3 e^{-3x} - \frac{4}{9}x^2 e^{-3x} - \frac{8}{27}x e^{-3x} - \frac{8}{81}e^{-3x} + C}$$

6. $\int e^{3x}\cos 2x \, dx$

$u = e^{3x}$ $\quad dv = \cos 2x \, dx$

$du = 3e^{3x}\,dx$ $\quad v = \frac{1}{2}\sin 2x$

$$\frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\int e^{3x}\sin 2x \, dx$$

$u = e^{3x}$ $\quad dv = \sin 2x \, dx$

$du = 3e^{3x}\,dx$ $\quad v = \frac{-1}{2}\cos 2x$

$$\frac{1}{2}e^{3x}\sin 2x - \frac{3}{2}\left(\frac{-1}{2}e^{3x}\cos 2x + \frac{3}{2}\int e^{3x}\cos 2x \, dx\right)$$

$$Q = \frac{1}{2}e^{3x}\sin 2x + \frac{3}{4}e^{3x}\cos 2x - \frac{9}{4}Q$$

$$Q = \boxed{\frac{2}{13}e^{3x}\sin 2x + \frac{3}{13}e^{3x}\cos 2x + C}$$

7. $\int \cot^{-1} 3x \, dx$

$u = \cot^{-1} 3x$ $\quad dv = dx$

$du = \frac{-3}{1 + 9x^2}\,dx$ $\quad v = x$

$$x\cot^{-1} 3x + \int \frac{3x}{1 + 9x^2}\,dx$$

$w = 1 + 9x^2$ $\quad dw = 18x\,dx$

or $\frac{1}{6}dw = 3x\,dx$ so

$$x\cot^{-1} 3x + \frac{1}{6}\int \frac{1}{w}\,dw \quad \text{and}$$

$$\boxed{x\cot^{-1} 3x + \frac{1}{6}\ln(1 + 9x^2) + C}$$

8. $\int \cos(\ln x)\,dx$

$w = \ln x$ $\quad e^w = x$

$e^w\,dw = dx$ $\quad \int e^w \cos w \, dw$

$u = e^w$ $\quad dv = \cos w \, dx$

$du = e^w\,dx$ $\quad v = \sin w$

$$e^w \sin w - \int e^w \sin w \, dw$$

$u = e^w$ $\quad dv = \sin w \, dx$

$du = e^w\,dx$ $\quad v = -\cos w$

$$e^w \sin w - \left(-e^w\cos w + \int e^w \cos w \, dw\right)$$

$$Q = e^w \sin w - (-e^w \cos w + Q)$$

$$Q = \boxed{\frac{1}{2}x\sin(\ln x) + \frac{1}{2}x\cos(\ln x) + C}$$

9. $\int x^5 e^{-x^2}\,dx$

$w = x^2$ $\quad \frac{1}{2}dw = x\,dx$

so $\int (x^2)^2 e^{-x^2}\,x\,dx$

$$\int \frac{1}{2}w^2 e^{-w}\,dw$$

Tabular Integration

u	dv
$\frac{1}{2}w^2$	e^{-w}
w	$-e^{-w}$
1	e^{-w}
0	$-e^{-w}$

$$\boxed{-\frac{1}{2}x^4 e^{-x^2} - x^2 e^{-x^2} - e^{-x^2} + C}$$

10. $\displaystyle\int_0^{\frac{\pi}{2}} x \sin 2x \, dx$

$\int x \sin 2x \, dx \qquad u = x \qquad du = dx$

$dv = \sin 2x \, dx \qquad v = \frac{-1}{2} \cos 2x$

$\frac{-1}{2} x \cos 2x + \frac{1}{2} \int \cos 2x \, dx$

$= \frac{-1}{2} x \cos 2x + \frac{1}{4} \sin 2x$ so

$\left[\frac{-1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}}$

$= \boxed{\frac{\pi}{4}}$

11. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^3 x \, dx$

$dv = \sec^2 x \, dx \qquad v = \tan x$

$u = \sec x \qquad du = \sec x \tan x \, dx$

$\int \sec^3 x \, dx \quad \rightarrow \quad \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \ln \left| \sec x + \tan x \right|$

$Q = \sec x \tan x - Q + \ln | \sec x + \tan x |$

$Q = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \left| \sec x + \tan x \right|$

$\left[\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \left| \sec x + \tan x \right| \right]_0^{\frac{\pi}{4}} = \boxed{\frac{\sqrt{2}}{2} + \frac{1}{2} \ln\left(1 + \sqrt{2}\right)}$