6.3 Integration by Parts Worksheet

$\int u\,dv = uv - \int v\,du$ The "trick" for using this technique correctly is to choose the dv properly – dv should be the derivative of something times dx. The dv should also take up as much as possible of the integrand.

Use u = L ogarithmic, I nverse Trigonometric, P olynomial, E xponential, T rigonometric

Evaluate the integrals below.

1. $\int \cos^{-1} x\,dx$

(Hint: $u = \cos^{-1} x$, $dv = dx$)

$u = \cos^{-1} x \qquad v = x$

$du = \dfrac{-1}{\sqrt{1-x^2}}\,dx \qquad dv = dx$

$= x\cos^{-1} x + \int \dfrac{x}{\sqrt{1-x^2}}\,dx$ and

$w = 1 - x^2 \rightarrow dw = -2x\,dx$ or $-\dfrac{1}{2}dw = x\,dx$

$= x\cos^{-1} x - \dfrac{1}{2}\int w^{\frac{-1}{2}}\,dw$

$= x\cos^{-1} x - w^{\frac{1}{2}} + C$ or

$= \boxed{x\cos^{-1} x - \sqrt{1-x^2} + C}$

2. $\int \dfrac{x}{3}\csc x \cot x\,dx$

(Hint: $u = \dfrac{x}{3}$)

$u = \dfrac{x}{3} \qquad v = -\csc x$

$du = \dfrac{1}{3}dx \qquad dv = \csc x \cot x\,dx$

$= \dfrac{-x}{3}\csc x + \dfrac{1}{3}\int \csc x\,dx$

$= \dfrac{-x}{3}\csc x + \dfrac{1}{3}\int \csc x\,\dfrac{(\csc x - \cot x)}{(\csc x - \cot x)}\,dx$

and denominator is $w = \csc x - \cot x$ so

numerator is $dw = (-\csc x \cot x + \csc^2 x)\,dx$

$= \boxed{\dfrac{-x}{3}\csc x + \dfrac{1}{3}\ln\left|\csc x - \cot x\right| + C}$

3. $\int_0^1 \cos^{-1} x\,dx$

(Hint: use the antiderivative that you found in 1, above)

$\left(1\cos^{-1} 1 - \sqrt{1-1^2}\right) - \left(0\cos^{-1} 0 - \sqrt{1-0^2}\right)$

$= \boxed{1}$

4. $\int x^3 \sin\left(\dfrac{x}{2}\right)dx$

(Hint: Tabular Integration)

u	dv
x^3	$\sin\left(\frac{x}{2}\right)$
$3x^2$	$-2\cos\left(\frac{x}{2}\right)$
$6x$	$-4\sin\left(\frac{x}{2}\right)$
6	$8\cos\left(\frac{x}{2}\right)$
0	$16\sin\left(\frac{x}{2}\right)$

$\rightarrow$

$$\boxed{-2x^3\cos\left(\frac{x}{2}\right) + 12x^2\sin\left(\frac{x}{2}\right) + 48x\cos\left(\frac{x}{2}\right) - 96\sin\left(\frac{x}{2}\right) + C}$$

5. $\int_0^{\pi} x^3 \sin\left(\dfrac{x}{2}\right)dx$

(Hint: use the antiderivative that you found in 4, above)

6. $\int \cos\sqrt{x}\,dx$

(Hint: do a w – substitution first, $w = \sqrt{x}$, $x = w^2$, $dx = ?$)

$(0 + 12\pi^2(1) + 0 - 96(1)) - (0)$

$= \boxed{12\pi^2 - 96}$

$w = \sqrt{x}$ so $x = w^2$ and $dx = 2w\,dw$

$\rightarrow \int 2w\cos w\,dw$ Tabular Integration

u	dv
$2w$	$\cos w$
2	$\sin w$
0	$-\cos w$

$\rightarrow 2w\sin w + 2\cos w + C$

OR

$\boxed{2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C}$

7. $\int e^{-3x}\cos\frac{x}{2}\,dx$

(Hint: Do integration by parts twice. When the original integral reappears, replace it with A. Solve for A)

$u = e^{-3x}$ $\quad v = 2\sin\frac{x}{2}$

$du = -3e^{-3x}\,dx$ $\quad dv = \cos\frac{x}{2}$

$= 2e^{-3x}\sin\frac{x}{2} + 6\int e^{-3x}\sin\frac{x}{2}\,dx$ and

$u = e^{-3x}$ $\quad v = -2\cos\frac{x}{2}$

$du = -3e^{-3x}\,dx$ $\quad dv = \sin\frac{x}{2}$

$= 2e^{-3x}\sin\frac{x}{2} + 6\left(-2e^{-3x}\cos\frac{x}{2} - 6\int e^{-3x}\cos\frac{x}{2}\,dx\right)$

so $A = 2e^{-3x}\sin\frac{x}{2} - 12e^{-3x}\cos\frac{x}{2} - 36A$

$37A = 2e^{-3x}\sin\frac{x}{2} - 12e^{-3x}\cos\frac{x}{2}$ and

$A = \boxed{\frac{2}{37}e^{-3x}\sin\frac{x}{2} - \frac{12}{37}e^{-3x}\cos\frac{x}{2} + C}$

8. $\int_0^{\pi} e^{-3x}\cos\frac{x}{2}\,dx$

(Hint: use the antiderivative found in 7, to the left)

$\left(\frac{2}{37}e^{-3\pi}\sin\frac{\pi}{2} - \frac{12}{37}e^{-3\pi}\cos\frac{\pi}{2}\right)$

$- \left(\frac{2}{37}e^{-3(0)}\sin\frac{0}{2} - \frac{12}{37}e^{-3(0)}\cos\frac{0}{2}\right)$

$= \boxed{\frac{2}{37}e^{-3\pi} + \frac{12}{37}}$

9. $\int \csc^3 x\,dx$

(Hint: choose a dv so that a v is readily apparent)

$u = \csc x$ $\quad v = -\cot x$

$du = -\csc x\cot x\,dx$ $\quad dv = \csc^2 x\,dx$

$= -\csc x\cot x - \int \csc x\cot^2 x\,dx$

$= -\csc x\cot x - \int \csc x(\csc^2 x - 1)\,dx$

$= -\csc x\cot x - \int \csc^3 x\,dx + \int \csc x\,dx$ or

$A = -\csc x\cot x - A + \ln\left|\csc x - \cot x\right|$

$A = \boxed{\frac{1}{2}\csc x\cot x + \frac{1}{2}\left|\csc x - \cot x\right| + C}$

10. $\int \cos(\ln x^3)\,dx$

$= \int \cos(3\ln x)\,dx$

$w = 3\ln x$ so $e^{\frac{w}{3}} = x$

and $dx = \frac{1}{3}e^{\frac{w}{3}}\,dw$

so $\rightarrow \frac{1}{3}\int e^{\frac{w}{3}}\cos w\,dw$

and using Integration by Parts

Twice, the integral reappears,

and we get

$\boxed{\frac{1}{10}x\cos(\ln x^3) + \frac{3}{10}x\sin(\ln x^3) + C}$

11. $\int \log_4 x^2\,dx$

$= \frac{2}{\ln 4}\int \ln x\,dx$

and $w = \ln x$

so $e^w = x$ $\quad dx = e^w\,dw$

$= \frac{2}{\ln 4}\int w\,e^w\,dw$

u	dv
w	e^w
1	e^w
0	e^w

$= \frac{2}{\ln 4}(w\,e^w - e^w) + C$

$= \boxed{\frac{2}{\ln 4}(x\ln x - x) + C}$