6.4 Exponential Growth and Decay

Growth and Decay Equations $\frac{dy}{dt} = ky$ and $y = y_0 e^{kt}$

$$\int \frac{1}{y}\,dy = \int k\,dt \quad \rightarrow \quad \ln y = kt + C \qquad y = e^{kt+C} = e^C e^{kt} = y_0 e^{kt}$$

Money compounding k times a year, where r is the rate $A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}$

Money compounding continuously $A(t) = A_0 e^{rt}$

Newton′s Law of Cooling, $\frac{dT}{dt} = -k(T - T_s)$ and $T - T_s = (T_0 - T_s)e^{-kt}$
where T_0 is the initial temperature, and T_s is the surrounding temperature

Motion of a Coasting Object, $\frac{dv}{dt} = \frac{-k}{m}v$, and $v = v_0 e^{-\left(\frac{k}{m}\right)t}$ where $k > 0$, v_0 is the initial velocity, and m is the mass.

Solve the following differential equations in 1 and 2.

1. $\frac{dy}{dx} = x^2 e^{-y} \sec e^y$

$e^y \cos e^y\,dy = x^2\,dx$ so

$$\int e^y \cos e^y\,dy = \int x^2\,dx$$

left $\rightarrow$ $u = e^y$ $du = e^y\,dy$

$$\int \cos u\,du = \sin e^y = \frac{1}{3}x^3 + C \quad \text{so}$$

$$e^y = \sin^{-1}\left(\frac{1}{3}x^3 + C\right)$$

$$\boxed{y = \ln\left(\sin^{-1}\left(\frac{1}{3}x^3 + C\right)\right)}$$

2. $\frac{dy}{dx} = \frac{(1 + y^2)^2}{xy\ln x}$

$$\frac{y}{(1 + y^2)^2}\,dy = \frac{1}{x\ln x}\,dx$$

$$\int \frac{y}{(1 + y^2)^2}\,dy = \int \frac{1}{x\ln x}\,dx$$

left $\rightarrow$ $u = 1 + y^2$ $du = 2y\,dy$ or $\frac{1}{2}du = y\,dy$

right $\rightarrow$ $w = \ln x$ $dw = \frac{1}{x}dx$ so

$$\frac{1}{2}\int u^{-2}\,du = \int \frac{1}{w}\,dw \quad \text{so}$$

$$\frac{-1}{2u} = \ln|w| + C \quad \text{or} \quad \boxed{\frac{-1}{2(1 + y^2)} = \ln|\ln x| + C}$$

Find the solution to $\frac{dy}{dt} = ky$, with the given initial conditions in 3 and 4.

3. $k = 0.5$, $y(0) = 200$

$y = y_0 e^{kt}$ $y = 200\left(e^{\frac{1}{2}}\right)^t = \boxed{200\,e^{\frac{t}{2}}}$

4. $y(0) = 80$, $y(20) = 400$

$400 = 80(e^k)^{20}$ $5 = (e^k)^{20}$ $e^k = 5^{\frac{1}{20}}$

$y = \boxed{(80)\,5^{\frac{t}{20}}}$

For an investment compounding continuously, find the missing values in 5 and 6.

5. If the initial deposit is \$300, and it takes 10 years for the investment to double in value, how much money will be in the account after 30 years?

$A = A_0 e^{rt}$ $600 = 300(e^r)^{10}$ $2 = (e^r)^{10}$

$e^r = 2^{\frac{1}{10}}$ $A = 300\left(2^{\frac{t}{10}}\right)$ $A = 300(2^3) = \boxed{\$2400}$

6. If the annual rate is 6.5 %, and the amount of money after 20 years will be \$20,000, find the initial amount in the account, and how long it will take for the investment to double

$20000 = A_0(e^{0.065})^{20}$ $A_0 = \frac{20000}{e^{1.3}} \approx \boxed{\$5450.64}$

$A = 5450.65(e^{0.065})^t$ $2 = (e^{0.065})^t$ $\boxed{t = \frac{\ln 2}{0.065}\text{ yrs}}$

7. Radium has a half – life of approximately 1600 years. Find a formula for the amount of radium remaining if the initial amount is 50 milligrams. Find when there will be 40 milligrams left.

$$\frac{1}{2} = (e^k)^{1600} \quad \text{so} \quad e^k = \left(\frac{1}{2}\right)^{\frac{1}{1600}} \quad \text{and}$$

$$\boxed{y = 50\left(\frac{1}{2}\right)^{\frac{t}{1600}}} \quad \text{so} \quad 40 = 50\left(\frac{1}{2}\right)^{\frac{t}{1600}} \quad \text{or}$$

$$\frac{4}{5} = \left(\frac{1}{2}\right)^{\frac{t}{1600}} \rightarrow \ln\left(\frac{4}{5}\right) = \frac{t}{1600}\ln\left(\frac{1}{2}\right) \quad \text{so}$$

$$t = \frac{1600\,(\ln 4 - \ln 5)}{-\ln 2} \text{ years} \quad \text{or}$$

$$\boxed{t = \frac{1600\,(\ln 5 - \ln 4)}{\ln 2} \text{ years}}$$

8. A cup of coffee is placed in the microwave and brought to a temperature of 125°. The cup is removed from the microwave and placed on the counter, where the room temperature is 75°. Find a formula for the temperature of the cup if it cools to 100° in half an hour. Find the temperature at the end of another half an hour

$$T - T_s = (T_0 - T_s)e^{-kt}$$

$$\rightarrow \quad 100 - 75 = (125 - 75)(e^{-k})^{\frac{1}{2}}$$

$$\frac{1}{2} = (e^{-k})^{\frac{1}{2}} \rightarrow e^{-k} = \frac{1}{4}$$

$$\boxed{T = 75 + 50\left(\frac{1}{4}\right)^t} \qquad T = 75 + 50\left(\frac{1}{4}\right)^1 = \boxed{87.5°}$$

9. Veterinarians use sodium pentobarbital to anesthetize animals. Suppose that to anesthetize a dog, 30 milligrams are required for each kilogram of body weight. If the drug is eliminated exponentially from the bloodstream, and half is eliminated in 4 hours, approximate the single dose that will anesthetize a 20 kg dog for 45 minutes.

$$y = y_0 e^{kt} \qquad 600 = y_0 (e^k)^{\frac{3}{4}} \qquad \frac{1}{2} = (e^k)^4 \qquad e^k = \left(\frac{1}{2}\right)^{\frac{1}{4}} \qquad 600 = y_0\left(\left(\frac{1}{2}\right)^{\frac{1}{4}}\right)^{\frac{3}{4}}$$

$$600 = y_0\left(\frac{1}{2}\right)^{\frac{3}{16}} \qquad y_0 = \frac{600}{\left(\frac{1}{2}\right)^{\frac{3}{16}}} = 600\left(2^{\frac{3}{16}}\right) \text{ milligrams} \approx \boxed{683.273 \text{ milligrams}}$$