7.3 Volume Of A Solid With Known Cross Sections

The volume of a solid of known integrable cross section area $A(x)$ from $x = a$ to $x = b$ can be represented by

$$V = \int_a^b A(x)\,dx$$

Steps for finding the volume of a solid with known cross sections.

(1) Sketch the solid and a typical cross section.

(2) Find a formula for $A(x)$.

(3) Find the limits of integration.

(4) Evaluate the integral $V = \int_a^b A(x)\,dx$

For problems 1 – 6, find the volume of the indicated solid.

1. A solid has as its base the region in the xy – plane bounded by the graphs of $x = 4$ and $x = y^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x – axis is a square with a base in the xy – plane.

$$V = \int_0^4 s^2\,dx = \int_0^4 \left(2\sqrt{x}\right)^2 dx$$

$$V = \int_0^4 4x\,dx = \left[2x^2\right]_0^4 = \boxed{32}$$

2. A solid has as its base the region in the xy – plane bounded by the graph of $x^2 + y^2 = 9$. Find the volume of the solid if every cross section by a plane perpendicular to the x – axis is an equilateral triangle with base in the xy – plane.

$$V = 2\int_0^3 \frac{\sqrt{3}}{4}\,s^2\,dx$$

$$V = \frac{\sqrt{3}}{2}\int_0^3 \left(2\sqrt{9 - x^2}\right)^2 dx$$

$$V = 2\sqrt{3}\int_0^3 (9 - x^2)\,dx = 2\sqrt{3}\left[9x - \frac{x^3}{3}\right]_0^3$$

$$V = 2\sqrt{3}\,(27 - 9) = \boxed{36\sqrt{3}}$$

3. A solid has as its base the region in the xy – plane bounded by the graph $y = \sin x$ and the x – axis on the interval $[0, \pi]$. Find the volume of the solid if every cross section by a plane perpendicular to the x – axis is a circle with a diameter in the xy – plane.

$$V = \int_0^\pi \pi r^2\,dx = \pi\int_0^\pi \left(\tfrac{1}{2}\sin x\right)^2 dx$$

$$V = \frac{\pi}{4}\int_0^\pi \sin^2 x\,dx = \frac{\pi}{8}\int_0^\pi (1 - \cos 2x)\,dx$$

$$V = \frac{\pi}{8}\left[x - \tfrac{1}{2}\sin 2x\right]_0^\pi = \boxed{\frac{\pi^2}{8}}$$

4. A solid has as its base the region in the xy – plane bounded by the graph of $x^2 + y^2 = 4$. Find the volume of the solid if every cross section by a plane perpendicular to the y – axis is a rectangle with a base in the xy – plane, and height equal to $\frac{1}{2}$ the base.

$$V = 2\int_0^2 b\,h\,dy$$

$$V = 2\int_0^2 \left(2\sqrt{4 - y^2}\right)\left(\sqrt{4 - y^2}\right) dy$$

$$V = 4\int_0^2 (4 - y^2)\,dy = 4\left[4y - \frac{y^3}{3}\right]_0^2 \quad \text{so}$$

$$V = 4\left(8 - \frac{8}{3}\right) = \boxed{\frac{64}{3}}$$

5. A solid has as its base the region in the xy – plane bounded by the graphs of $y = 9$ and $y = x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x – axis is a semicircle with a diameter in the xy – plane.

$$V = 2\int_0^3 \frac{\pi}{2} r^2\, dx = \pi \int_0^3 \left(\frac{9-x^2}{2}\right)^2 dx$$

$$V = \frac{\pi}{4}\int_0^3 (81 - 18x^2 + x^4)\, dx \quad \text{so}$$

$$V = \frac{\pi}{4}\left[81x - \frac{18}{3}x^3 + \frac{x^5}{5}\right]_0^3 \quad \text{so}$$

$$V = \frac{\pi}{4}\left(243 - 162 + \frac{243}{5}\right) = \boxed{\frac{162\pi}{5}}$$

6. A solid has as its base the region in the xy – plane bounded by the graphs of $y = x$ and $y = x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the y – axis is a square with a diagonal in the xy – plane.

$$V = \int_0^1 s^2\, dy = \int_0^1 \left(\frac{\sqrt{y} - y}{\sqrt{2}}\right)^2 dy$$

$$V = \frac{1}{2}\int_0^1 \left(y - 2y^{\frac{3}{2}} + y^2\right) dy \quad \text{so}$$

$$V = \frac{1}{2}\left[\frac{y^2}{2} - \frac{4}{5}y^{\frac{5}{2}} + \frac{y^3}{3}\right]_0^1 \quad \text{so}$$

$$V = \frac{1}{2}\left(\frac{1}{2} - \frac{4}{5} + \frac{1}{3}\right) = \boxed{\frac{1}{60}}$$