7.4 Lengths of Curves

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

if y is a smooth function of x on [a, b].

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

if x is a smooth function of y on [c, d].

Here's a proof of the first formula:

Suppose that a smooth curve is randomly partitioned on the interval [a, b]. Then we can approximate the length of the curve by $L_P = \sum_{k=1}^{n} d(Q_{k-1}, Q_k)$ where $d(Q_{k-1}, Q_k)$ represents the distance between two points on the partition, or $d(Q_{k-1}, Q_k) = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$ and $\Delta x_k = x_k - x_{k-1}$ so

$d(Q_{k-1}, Q_k) = \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$ Also, the Mean Value Theorem guarantees that

$f(x_k) - f(x_{k-1}) = f'(w_k)(x_k - x_{k-1})$ for some w_k on $[x_k, x_{k-1}]$ so

$d(Q_{k-1}, Q_k) = \sqrt{(\Delta x_k)^2 + (f'(w_k)(x_k - x_{k-1}))^2} = \sqrt{(\Delta x_k)^2 + (f'(w_k)\Delta x_k)^2}$ or

$d(Q_{k-1}, Q_k) = \Delta x_k \sqrt{1 + (f'(w_k))^2}$ and so $L_P = \sum_{k=1}^{n} \Delta x_k \sqrt{1 + (f'(w_k))^2}$ and we can find the exact length of the curve with $L = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta x_k \sqrt{1 + (f'(w_k))^2}$ on [a, b], or $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

For problems 1 – 4, find the length of the curve from A to B.

1. $y = \frac{2}{3} x^{\frac{2}{3}}$ $A\left(1, \frac{2}{3}\right)$ $B\left(8, \frac{8}{3}\right)$

$\frac{dy}{dx} = \frac{4}{9} x^{\frac{-1}{3}}$ so $L = \int_1^8 \sqrt{1 + \left(\frac{4}{9x^{\frac{1}{3}}}\right)^2}\, dx$

$= \int_1^8 \sqrt{\frac{81x^{\frac{2}{3}}}{81x^{\frac{2}{3}}} + \frac{16}{81x^{\frac{2}{3}}}}\, dx$

$= \int_1^8 \frac{1}{9} x^{\frac{-1}{3}} \sqrt{81x^{\frac{2}{3}} + 16}\, dx$

$u = 81x^{\frac{2}{3}} + 16$ $du = 54x^{\frac{-1}{3}}\, dx = 486\left(\frac{1}{9} x^{\frac{-1}{3}}\, dx\right)$

so $L = \frac{1}{486} \int_{97}^{340} u^{\frac{1}{2}}\, du = \frac{1}{486}\left(\frac{2}{3}\right)\left[u^{\frac{3}{2}} \right]_{97}^{340}$

$= \boxed{\frac{1}{729}\left(340\sqrt{340} - 97\sqrt{97}\right)}$

2. $(y + 1)^2 = (x - 4)^3$ A(5, 0) B(8, 7)

$y = (x - 4)^{\frac{3}{2}} - 1$ $\frac{dy}{dx} = \frac{3}{2}(x - 4)^{\frac{1}{2}}$

$L = \int_5^8 \sqrt{1 + \left(\frac{3}{2}(x - 4)^{\frac{1}{2}}\right)^2}\, dx$

$= \int_5^8 \sqrt{1 + \frac{9}{4}(x - 4)}\, dx = \int_5^8 \sqrt{\frac{9}{4}x - 8}\, dx$

$u = \frac{9}{4}x - 8$ $\frac{4}{9}du = dx$

$\rightarrow \frac{4}{9} \int_{\frac{13}{4}}^{10} u^{\frac{1}{2}}\, du$

$= \frac{8}{27}\left[u^{\frac{3}{2}} \right]_{\frac{13}{4}}^{10} = \boxed{\frac{8}{27}\left(10\sqrt{10} - \frac{13}{8}\sqrt{13}\right)}$

3. $y = \frac{1}{4x} + \frac{x^3}{3}$ $\quad A\left(2, \frac{67}{24}\right)$ $\quad B\left(3, \frac{109}{12}\right)$

$$\frac{dy}{dx} = \frac{-1}{4x^2} + x^2 \quad \text{so} \quad L = \int_2^3 \sqrt{1 + \left(\frac{-1}{4x^2} + x^2\right)^2}\, dx$$

$$= \int_2^3 \sqrt{1 + \left(\frac{1}{16x^4} - \frac{1}{2} + x^4\right)}\, dx$$

$$= \int_2^3 \sqrt{\left(\frac{1}{16x^4} + \frac{1}{2} + x^4\right)}\, dx = \int_2^3 \sqrt{\left(\frac{1}{4x^2} + x^2\right)^2}\, dx$$

$$= \int_2^3 \left(\frac{1}{4}x^{-2} + x^2\right) dx = \left[\frac{-1}{4x} + \frac{x^3}{3}\right]_2^3$$

$$= \left(\frac{-1}{12} + 9\right) - \left(\frac{-1}{8} + \frac{8}{3}\right) = \boxed{\frac{51}{8}}$$

4. $30xy^3 - y^8 = 15$ $\quad A\left(\frac{8}{15}, 1\right)$ $\quad B\left(\frac{271}{240}, 2\right)$

$$30xy^3 = y^8 + 15 \quad \rightarrow \quad x = \frac{1}{30}y^5 + \frac{1}{2}y^{-3}$$

$$\text{so} \quad \frac{dx}{dy} = \frac{1}{6}y^4 - \frac{3}{2}y^{-4} \quad \text{and}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{6}y^4 - \frac{3}{2}y^{-4}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{1 + \frac{1}{36}y^8 - \frac{1}{2} + \frac{9}{4}y^{-8}}\, dy \quad \text{so}$$

$$L = \int_1^2 \sqrt{\frac{1}{36}y^8 + \frac{1}{2} + \frac{9}{4}y^{-8}}\, dy$$

$$= \int_1^2 \sqrt{\left(\frac{1}{6}y^4 + \frac{3}{2}y^{-4}\right)^2}\, dy$$

$$= \int_1^2 \left(\frac{1}{6}y^4 + \frac{3}{2}y^{-4}\right) dy$$

$$\text{so} \quad L = \left[\frac{1}{30}y^5 - \frac{1}{2}y^{-3}\right]_1^2$$

$$= \left(\frac{32}{30} - \frac{1}{16}\right) - \left(\frac{1}{30} - \frac{1}{2}\right) = \boxed{\frac{353}{240}}$$