7.4 Lengths of Curves

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

if y is a smooth function of x on $[a, b]$.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

if x is a smooth function of y on $[c, d]$.

Here's a proof of the first formula:

Suppose that a smooth curve is randomly partitioned on the interval [a, b]. Then we can approximate the length of the curve by $L_P = \sum_{k=1}^{n} d(Q_{k-1}, Q_k)$ where $d(Q_{k-1}, Q_k)$ represents the distance between two points on the partition,

or $d(Q_{k-1}, Q_k) = \sqrt{(x_k - x_{k-1})^2 + (f(x_k) - f(x_{k-1}))^2}$ and $\Delta x_k = x_k - x_{k-1}$ so

$d(Q_{k-1}, Q_k) = \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$ Also, the Mean Value Theorem guarantees that

$f(x_k) - f(x_{k-1}) = f'(w_k)(x_k - x_{k-1})$ for some w_k on $[x_k, x_{k-1}]$ so

$d(Q_{k-1}, Q_k) = \sqrt{(\Delta x_k)^2 + (f'(w_k)(x_k - x_{k-1}))^2} = \sqrt{(\Delta x_k)^2 + (f'(w_k)\Delta x_k)^2}$ or

$d(Q_{k-1}, Q_k) = \Delta x_k \sqrt{1 + (f'(w_k))^2}$ and so $L_P = \sum_{k=1}^{n} \Delta x_k \sqrt{1 + (f'(w_k))^2}$ and we can find the exact length of

the curve with $L = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \Delta x_k \sqrt{1 + (f'(w_k))^2}$ on $[a, b]$, or $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

For problems 1 – 4, find the length of the curve from A to B.

1. $y = \frac{2}{3}x^{\frac{2}{3}}$ $A\left(1, \frac{2}{3}\right)$ $B\left(8, \frac{8}{3}\right)$

2. $(y + 1)^2 = (x - 4)^3$ $A(5, 0)$ $B(8, 7)$

3. $y = \frac{1}{4x} + \frac{x^3}{3}$ $A\left(2, \frac{67}{24}\right)$ $B\left(3, \frac{109}{12}\right)$

4. $30xy^3 - y^8 = 15$ $A\left(\frac{8}{15}, 1\right)$ $B\left(\frac{271}{240}, 2\right)$