8.1 L′Hopital′s Rule

Suppose f and g are differentiable on an open interval (a, b) containing c, except possibly at c itself. If $\frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = c$ and if $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}, \quad \text{provided either} \quad \lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists} \quad \text{or} \quad \lim_{x \to c} \frac{f'(x)}{g'(x)} = \infty$$

This rule will allow us to find the limit of a variety of forms, including:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty \cdot 0,\ \infty - \infty,\ 1^{\infty},\ 0^{0},\ \infty^{0}$$

Solve problems 1 – 10.

1. $\lim_{x \to 0} \frac{\cos x + 4x - 1}{3x}$

$\rightarrow \frac{0}{0} \quad L' \quad \rightarrow$

$\lim_{x \to 0} \frac{-\sin x + 4}{3} = \boxed{\frac{4}{3}}$

2. $\lim_{x \to 1} \frac{1 - x + \ln x}{1 + \cos(\pi x)}$

$\rightarrow \frac{0}{0} \quad L' \quad \rightarrow$

$\lim_{x \to 1} \frac{-1 + \frac{1}{x}}{-\pi \sin(\pi x)} \rightarrow \frac{0}{0} \quad L' \quad \rightarrow$

$\lim_{x \to 1} \frac{\frac{-1}{x^2}}{-\pi^2 \cos(\pi x)} \rightarrow \boxed{\frac{-1}{\pi^2}}$

3. $\lim_{x \to \infty} \frac{e^x}{x^2 + x}$

$\rightarrow \frac{\infty}{\infty} \quad L' \quad \rightarrow$

$\lim_{x \to \infty} \frac{e^x}{2x + 1} \rightarrow \frac{\infty}{\infty} \quad L' \quad \rightarrow$

$\lim_{x \to \infty} \frac{e^x}{2} = \boxed{\infty}$

4. $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

$\rightarrow \frac{\infty}{\infty} \quad L' \quad \rightarrow$

$\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = \boxed{0}$

5. $\lim_{x \to \infty} x \ln\left(\frac{x - 1}{x + 1}\right)$

$\rightarrow \infty(0) = \lim_{x \to \infty} \frac{\ln\left(\frac{x-1}{x+1}\right)}{\frac{1}{x}} \rightarrow \frac{0}{0}$

$L' \rightarrow \lim_{x \to \infty} \frac{\frac{x+1}{x-1}\left(\frac{(x+1)-(x-1)}{(x+1)^2}\right)}{\frac{-1}{x^2}}$

$= \lim_{x \to \infty} \frac{\frac{2}{x^2 - 1}}{\frac{-1}{x^2}} = \lim_{x \to \infty} \frac{-2x^2}{x^2 - 1} = \boxed{-2}$

6. $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

$\rightarrow \infty - \infty \quad \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \rightarrow \frac{0}{0}$

$L' \rightarrow \lim_{x \to 0^+} \frac{\cos x - 1}{x \cos x + \sin x} \rightarrow \frac{0}{0} \quad L' \rightarrow$

$\lim_{x \to 0^+} \frac{-\sin x}{-x \sin x + \cos x + \cos x} \rightarrow \frac{0}{2} = \boxed{0}$

7. $\lim_{x \to \infty} \left(\sqrt{x^2 + 5x} - x\right)$

8. $\lim_{x \to 0^+} (1 + 2x)^{\frac{1}{3x}}$

$\rightarrow \quad \infty - \infty$ so

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 5x} - x\right) \frac{\left(\sqrt{x^2 + 5x} + x\right)}{\left(\sqrt{x^2 + 5x} + x\right)} =$$

$$\lim_{x \to \infty} \frac{x^2 + 5x - x^2}{\left(\sqrt{x^2 + 5x} + x\right)} =$$

$$\lim_{x \to \infty} \frac{5x}{\left(\sqrt{x^2 + 5x} + x\right)} =$$

$$\lim_{x \to \infty} \frac{5}{\left(\sqrt{1 + \frac{5}{x}} + 1\right)} = \boxed{\frac{5}{2}}$$

$\rightarrow \quad 1^{\infty} \qquad f(x) = e^{\ln f(x)} \qquad f(x)^{g(x)} = e^{g(x) \ln f(x)}$

so $\displaystyle \lim_{x \to 0^+} e^{\frac{\ln(1+2x)}{3x}}$ FOCUS ON EXPONENT

$$\lim_{x \to 0^+} \frac{\ln(1 + 2x)}{3x} \rightarrow \frac{0}{0} \quad L' \rightarrow$$

$$\lim_{x \to 0^+} \frac{\frac{2}{1+2x}}{3} = \frac{2}{3} \rightarrow \boxed{e^{\frac{2}{3}}}$$

9. $\displaystyle \lim_{x \to 1^+} (x - 1)^{\ln x}$

$\rightarrow \quad 0^0$ so $\displaystyle \lim_{x \to 1^+} e^{(\ln x)\ln(x-1)}$

FOCUS ON EXPONENT $\displaystyle \lim_{x \to 1^+} (\ln x) \ln(x - 1)$

$$\rightarrow \quad 0(-\infty) \qquad \lim_{x \to 1^+} \frac{\ln(x - 1)}{\frac{1}{(\ln x)}} \rightarrow \frac{-\infty}{\infty}$$

$$L' \rightarrow \lim_{x \to 1^+} \frac{\frac{1}{x-1}}{\frac{-1}{(\ln x)^2}\left(\frac{1}{x}\right)}$$

$$\lim_{x \to 1^+} \frac{-x(\ln x)^2}{x - 1} \rightarrow \frac{0}{0} \quad L' \rightarrow$$

$$\lim_{x \to 1^+} \frac{-x\left(2\ln x\left(\frac{1}{x}\right)\right) - (\ln x)^2}{1} \rightarrow \frac{0}{1} = 0$$

so $e^0 = \boxed{1}$

10. $\displaystyle \lim_{x \to \infty} (\ln x)^{\frac{1}{x}}$

$\rightarrow \quad \infty^0$ so $\displaystyle \lim_{x \to \infty} e^{\frac{\ln(\ln x)}{x}}$

FOCUS ON EXPONENT $\displaystyle \lim_{x \to \infty} \frac{\ln(\ln x)}{x}$

$$\rightarrow \frac{\infty}{\infty} \quad L' \rightarrow \lim_{x \to \infty} \frac{\frac{1}{\ln x}\left(\frac{1}{x}\right)}{1} \rightarrow 0$$

so $e^0 = \boxed{1}$