8.2 Relative Rates of Growth

COMPARING RATES OF GROWTH

Let $f(x)$ and $g(x)$ be positive for x sufficiently large

1. f grows faster than g as $x \to \infty$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$

2. f grows slower than g as $x \to \infty$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

3. f grows at the same rate as g as $x \to \infty$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \neq 0$, where L is some real number

LITTLE – OH

Let $f(x)$ and $g(x)$ be positive for x sufficiently large

Then f is of smaller order than g as $x \to \infty$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$ and $f = o(g)$

BIG – OH

Let $f(x)$ and $g(x)$ be positive for x sufficiently large

Then f is of at most the order of g as $x \to \infty$ if there is a positive integer M for which $\frac{f(x)}{g(x)} \leq M$ for x sufficiently large, and $f = O(g)$

For problems 1 – 6, determine if $f(x)$ grows faster than, or slower than, or at the same rate as $g(x)$

1. $f(x) = \log_3 x$, $g(x) = \ln x^2$

$$\lim_{x \to \infty} \frac{\log_3 x}{2 \ln x} = \lim_{x \to \infty} \frac{\ln x}{2(\ln 3)(\ln x)} = \frac{1}{2 \ln 3}$$

so **f and g grow at the same rate**

2. $f(x) = \left(\frac{x}{2}\right)^2$, $g(x) = 10x - 1$

$$\lim_{x \to \infty} \frac{x^2}{4(10x - 1)} = \infty$$ so **f grows faster than g**

3. $f(x) = 2^x$, $g(x) = e^x$

$$\lim_{x \to \infty} \frac{2^x}{e^x} = \lim_{x \to \infty} \left(\frac{2}{e}\right)^x = 0$$

so **f grows slower than g**

4. $f(x) = 3^{2x}$, $g(x) = 3^x$

$$\lim_{x \to \infty} \frac{3^x(3^x)}{3^x} = \lim_{x \to \infty} 3^x = \infty$$

so **f grows faster than g**

5. $f(x) = \log \sqrt{x}$, $g(x) = \ln x$

$$\lim_{x \to \infty} \frac{\frac{1}{2} \log x}{\ln x} = \lim_{x \to \infty} \frac{\ln x}{2(\ln 10)(\ln x)} = \frac{1}{2 \ln 10}$$

so **f grows at the same rate as g**

6. $f(x) = x - \ln x^2$, $g(x) = \log_4 x$

$$\lim_{x \to \infty} \frac{x - 2 \ln x}{\log_4 x} = \lim_{x \to \infty} \frac{\ln 4 (x - 2 \ln x)}{\ln x} =$$

$$\lim_{x \to \infty} \frac{(\ln 4)x}{\ln x} - \lim_{x \to \infty} \frac{2(\ln 4)\ln x}{\ln x} =$$

$$\infty - 2(\ln 4) = \infty$$ so **f grows faster than g**

7. Order the following functions from slowest – growing to fastest – growing as $x \to \infty$, 2^x, x^x, $(\ln x)^x$, 2^{2x}

By inspection, $2^x < 4^x < (\ln x)^x < x^x$

For problems 8 – 15, determine whether the statement is true or false as $x \to \infty$

8. $x = o(3x)$

$$\lim_{x \to \infty} \frac{x}{3x} = \frac{1}{3}$$ **FALSE**

9. $x + \sin x = O\left(\frac{x}{2}\right)$

$$\lim_{x \to \infty} \frac{2(x + \sin x)}{x} = 2$$ **TRUE**

10. $4^x = O\left(4^{\frac{x}{2}}\right)$

11. $\sqrt{x} \log x = o(x)$

$$\lim_{x \to \infty} \frac{4^x}{2^x} = \lim_{x \to \infty} 2^x = \infty$$ **FALSE**

$$\lim_{x \to \infty} \frac{\sqrt{x}\,\log x}{x} = \lim_{x \to \infty} \frac{\log x}{\sqrt{x}} = 0$$ **TRUE**

12. $2^x + x^2 = O(2^x)$

$$\lim_{x \to \infty} \frac{2^x + x^2}{2^x} = \lim_{x \to \infty} \frac{2^x}{2^x} + \lim_{x \to \infty} \frac{x^2}{2^x} =$$

$1 + 0 = 1$ **TRUE**

13. $\frac{3}{x} = o(\ln x)$

$$\lim_{x \to \infty} \frac{\frac{3}{x}}{\ln x} = 0$$ **TRUE**

14. $\sqrt{x^2 + x} = O(2x)$

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + x}}{2x} = \frac{1}{2}$$ **TRUE**

15. $\ln(2x) = O((\ln x)^2)$

$$\lim_{x \to \infty} \frac{\ln 2 + \ln x}{(\ln x)(\ln x)} = 0$$ **TRUE**