8.3 Comparison Tests

1. Consider $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$

A. Converges for $0 < p < 1$

B. Diverges for $p \ge 1$

I. $p = 1 \quad \to \quad \displaystyle\lim_{b \to 0^+} \int_b^1 \frac{1}{x}\,dx = \lim_{b \to 0^+} \Big[\ln x\Big]_b^1 = \ln 1 - \lim_{b \to 0^+} \ln b = \infty$, Diverges

II. $0 < p < 1 \quad \displaystyle\lim_{b \to 0^+} \int_b^1 \frac{1}{x^p}\,dx = \lim_{b \to 0^+} \int_b^1 x^{-p}\,dx = \lim_{b \to 0^+} \Big[\frac{x^{1-p}}{1-p}\Big]_b^1$

$\displaystyle = \frac{1}{1-p} \lim_{b \to 0^+} \left(1 - b^{1-p}\right) = \frac{1}{1-p}$, Converges

III. $p > 1 \quad \displaystyle\lim_{b \to 0^+} \int_b^1 \frac{1}{x^p}\,dx = \lim_{b \to 0^+} \int_b^1 x^{-p}\,dx = \lim_{b \to 0^+} \Big[\frac{x^{1-p}}{1-p}\Big]_b^1$

$\displaystyle = \frac{1}{1-p} \lim_{b \to 0^+} \left(1 - b^{1-p}\right) = \frac{1-\infty}{1-p} \to \infty$, Diverges

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2. Consider $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$

A. Diverges for $0 < p \le 1$

B. Converges for $p > 1$

I. $p = 1 \quad \displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b \to \infty} \Big[\ln x\Big]_1^b = \lim_{b \to \infty} \ln b - 0 \to \infty$, Diverges

II. $0 < p < 1 \quad \displaystyle\lim_{b \to \infty} \int_1^b x^{-p}\,dx = \lim_{b \to \infty} \Big[\frac{1}{1-p}\, x^{1-p}\Big]_1^b$

$\displaystyle = \frac{1}{1-p}\left(\lim_{b \to \infty} b^{1-p} - 1\right) = \infty$, Diverges

III. $p > 1 \quad \displaystyle\lim_{b \to \infty} \int_1^b x^{-p}\,dx = \lim_{b \to \infty} \Big[\frac{1}{1-p}\, x^{1-p}\Big]_1^b$

$\displaystyle = \frac{1}{1-p}\left(\lim_{b \to \infty} b^{1-p} - 1\right) = \frac{0-1}{1-p} = \frac{1}{p-1}$, Converges

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DIRECT COMPARISON TEST

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges

LIMIT COMPARISON TEST

If the positive functions f and g are continuous on $[a, \infty)$ and if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$, $0 < L < \infty$

Then $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} g(x)\,dx$ both converge or both diverge

For problems 1 – 8, use integration, the direct comparison test, or the limit comparison test to determine whether the integral converges or diverges.

1. $\int_1^{\infty} \frac{3 + \cos x}{x^3}\,dx$

Direct → $3 + \cos x \le 4$

→ $\frac{3 + \cos x}{x^3} \le \frac{4}{x^3}$ → $4\int_1^{\infty} \frac{1}{x^3}\,dx$

and we know that this new integral converges

because $p = 3$, and because the original integral

is smaller, then it also **Converges**

Limit → $\lim_{x \to \infty} \frac{\frac{3 + \cos x}{x^3}}{\frac{4}{x^3}}$

$= \lim_{x \to \infty} \frac{(3 + \cos x)\,x^3}{4x^3} = \lim_{x \to \infty} \frac{(3 + \cos x)}{4}$

which Does not exist, so no conclusion

2. $\int_8^{\infty} \frac{1}{\sqrt[3]{x} - 1}\,dx$

Direct → $\sqrt[3]{x} - 1 \le \sqrt[3]{x}$

→ $\frac{1}{\sqrt[3]{x} - 1} \ge \frac{1}{\sqrt[3]{x}}$ and

$\int_8^{\infty} \frac{1}{x^{\frac{1}{3}}}\,dx$ with $p = \frac{1}{3}$ Diverges,

and the original is larger, so it must also **Diverge**

Limit → $\lim_{x \to \infty} \frac{\frac{1}{\sqrt[3]{x} - 1}}{\frac{1}{\sqrt[3]{x}}} = 1$

so they both Diverge

3. $\int_{-\infty}^{0} \frac{1}{x^2}\,dx$

4. $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt[4]{x} + \cos x}\,dx$

$= \int_{-\infty}^{-1} \frac{1}{x^2}\,dx + \int_{-1}^{0} \frac{1}{x^2}\,dx$

checking the first integral, $\int_{-\infty}^{-1} \frac{1}{x^2}\,dx$

$= \int_{1}^{\infty} \frac{1}{x^2}\,dx \rightarrow$ Converges

now the second integral, $\int_{-1}^{0} \frac{1}{x^2}\,dx$

$= \int_{0}^{1} \frac{1}{x^2}\,dx \rightarrow$ Diverges

So the original **Diverges**

$\rightarrow$ Direct $\rightarrow \sqrt[4]{x} + \cos x \geq \sqrt[4]{x}$

$\rightarrow \frac{1}{\sqrt[4]{x} + \cos x} \leq \frac{1}{\sqrt[4]{x}}$

$\int_{0}^{\frac{\pi}{2}} \frac{1}{x^{\frac{1}{4}}}\,dx \rightarrow p = \frac{1}{4}$ Converges,

so the original integral, which is smaller,

must also **Converge**

5. $\int_{-1}^{\infty} \left(\frac{1}{x+1} - \frac{1}{x+3} \right) dx$

$= \int_{-1}^{0} \left(\frac{1}{x+1} - \frac{1}{x+3} \right) dx + \int_{0}^{\infty} \left(\frac{1}{x+1} - \frac{1}{x+3} \right) dx$

checking the first integral, $\int_{-1}^{0} \left(\frac{1}{x+1} - \frac{1}{x+3} \right) dx$

$\rightarrow \lim_{b \to -1^+} \left[\ln \left| \frac{x+1}{x+3} \right| \right]_b^0$

$= -\ln 3 - \lim_{b \to -1^+} \ln \left| \frac{b+1}{b+3} \right|$

$= -\ln 3 - (-\infty) \rightarrow \infty$, **Diverges**

and there is no need to check the second integral

6. $\int_{e}^{\infty} \frac{2}{x - \ln x}\,dx$

Direct $\rightarrow x - \ln x \leq x$

$\frac{1}{x - \ln x} \geq \frac{1}{x}$ and $2\int_{e}^{\infty} \frac{1}{x}\,dx$

Diverges, so since the original is larger,

then it must also **Diverge**

Limit $\rightarrow \lim_{x \to \infty} \frac{\frac{1}{x - \ln x}}{\frac{1}{x}} = 1$ Both Diverge

7. $\int_{-\infty}^{0} \frac{4}{\sqrt{x^6 + 2}}\,dx$

8. $\int_{0}^{\infty} x^2 e^{-x}\,dx$

Direct $\rightarrow$ $\sqrt{x^6 + 2} \geq \sqrt{x^6}$

so $\dfrac{1}{\sqrt{x^6 + 2}} \leq \dfrac{1}{\sqrt{x^6}}$ and

$4 \displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{x^6}}\, dx = 4 \int_{0}^{\infty} \frac{1}{x^3}\, dx \rightarrow p = 3$

Converges, so the original integral, which is smaller, also Converges

Limit $\rightarrow$ $\displaystyle\lim_{x \to \infty} \frac{\frac{1}{\sqrt{x^6+2}}}{\frac{1}{\sqrt{x^6}}} = 1,$

so they both Converge

$= \displaystyle\lim_{b \to \infty} \int_0^b x^2 e^{-x}\, dx$ and

u	dv
x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$

so $\displaystyle\lim_{b \to \infty} \int_0^b x^2 e^{-x}\, dx$

$= \displaystyle\lim_{b \to \infty} \left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^b$

and $\displaystyle\lim_{b \to \infty} [(0 - 0 - 0) - (0 - 0 - 2)]$

$= 2,$ Converges