8.3 Improper Integrals

IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If $f(x)$ is continuous on $[a, \infty)$, then $$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then $$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

   where c is any real number

IMPROPER INTEGRALS WITH INFINITE DISCONTINUITIES

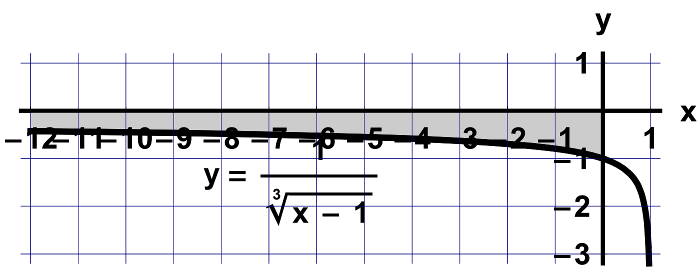
1. If $f(x)$ is continuous on $(a, b]$, then $$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

2. If $f(x)$ is continuous on $[a, b)$, then $$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

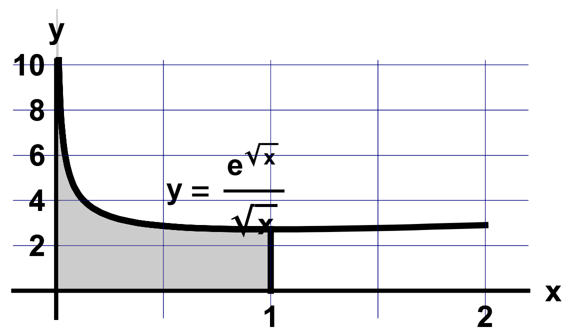
3. If $f(x)$ is continuous on $[a, c) \cup (c, b]$, then $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

For problems 1 – 8, evaluate the integral or state that it diverges

1. $$\int_{-\infty}^0 \frac{1}{\sqrt[3]{x-1}}\,dx$$



2. $$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$



$$= \lim_{b \to -\infty} \int_b^0 \frac{1}{\sqrt[3]{x-1}}\, dx$$

$\to \quad u = x - 1 \quad du = dx \quad$ so

$$\lim_{b \to -\infty} \int_{b-1}^{-1} u^{\frac{-1}{3}}\, du = \left(\frac{3}{2}\right) \lim_{b \to -\infty} \left[u^{\frac{2}{3}} \right]_{b-1}^{-1}$$

$$= \frac{3}{2} \lim_{b \to -\infty} \left(1 - (b-1)^{\frac{2}{3}} \right)$$

$$= \frac{3}{2}(1 - \infty) = \boxed{-\infty, \text{ Diverges}}$$

3. $\displaystyle\int_0^\infty \frac{x}{x^4 + 1}\, dx$

$$= \lim_{b \to \infty} \int_0^b \frac{x}{x^4 + 1}\, dx$$

$\to \quad u = x^2 \quad \frac{1}{2} du = x\, dx \quad$ so

$$\frac{1}{2} \lim_{b \to \infty} \int_0^{b^2} \frac{1}{u^2 + 1}\, du$$

$$= \frac{1}{2} \lim_{b \to \infty} \left[\tan^{-1} u \right]_0^{b^2}$$

$$= \frac{1}{2} \lim_{b \to \infty} (\tan^{-1} b^2 - 0)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{4}, \text{ Converges}}$$

5. $\displaystyle\int_{-3}^1 \frac{1}{x}\, dx$

$$= \int_{-3}^0 \frac{1}{x}\, dx + \int_0^1 \frac{1}{x}\, dx$$

$$\to \int_{-3}^0 \frac{1}{x}\, dx = \lim_{b \to 0^-} \int_{-3}^b \frac{1}{x}\, dx$$

$$= \lim_{b \to 0^-} \left[\ln|x| \right]_{-3}^{b}$$

$$= \lim_{b \to 0^-} (\ln|b| - \ln 3)$$

$$= -\infty - \ln 3 \to \boxed{-\infty, \text{ Diverges}}$$

$$= \lim_{b \to 0^+} \int_b^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

$\to \quad u = x^{\frac{1}{2}} \quad 2\, du = \frac{1}{\sqrt{x}} dx \quad$ so

$$2 \lim_{b \to 0^+} \int_{\sqrt{b}}^1 e^u\, du$$

$$= 2 \lim_{b \to 0^+} \left[e^u \right]_{\sqrt{b}}^{1} = 2 \lim_{b \to 0^+} \left(e - e^{\sqrt{b}} \right)$$

$$= \boxed{2(e - 1), \text{ Converges}}$$

4. $\displaystyle\int_{-2}^0 \frac{x}{\sqrt{4 - x^2}}\, dx$

$$= \lim_{b \to -2^+} \int_b^0 \frac{x}{\sqrt{4 - x^2}}\, dx$$

$\to \quad u = 4 - x^2 \quad \frac{-1}{2} du = x\, dx \quad$ so

$$\frac{-1}{2} \lim_{b \to -2^+} \int_{4-b^2}^4 u^{\frac{-1}{2}}\, du$$

$$= \frac{-1}{2} (2) \lim_{b \to -2^+} \left[u^{\frac{1}{2}} \right]_{4-b^2}^{4}$$

$$= - \lim_{b \to -2^+} \left(2 - (4 - b^2)^{\frac{1}{2}} \right)$$

$$-(2 - 0) = \boxed{-2, \text{ Converges}}$$

6. $\displaystyle\int_{-\infty}^\infty x e^{-x^2}\, dx$

$$= \int_{-\infty}^0 x e^{-x^2}\, dx + \int_0^\infty x e^{-x^2}\, dx$$

$$\to \int_0^\infty x e^{-x^2}\, dx = \lim_{b \to \infty} \int_0^b x e^{-x^2}\, dx$$

$u = -x^2 \quad -\frac{1}{2} du = x\, dx \quad$ so

$$\frac{-1}{2} \lim_{b \to \infty} \int_0^{-b^2} e^u\, du = \frac{-1}{2} \lim_{b \to \infty} \left[e^u \right]_0^{-b^2}$$

$$= \frac{-1}{2} \lim_{b \to \infty} \left(e^{-b^2} - 1\right) = \frac{1}{2}$$

using symmetry $\to \dfrac{-1}{2} + \dfrac{1}{2} = \boxed{0, \text{ Converges}}$

7. $$\int_0^{\infty} \frac{\cos x}{1 + \sin^2 x}\, dx$$

$$= \lim_{b \to \infty} \int_0^{b} \frac{\cos x}{1 + \sin^2 x}\, dx$$

$\rightarrow \quad u = \sin x \quad du = \cos x\, dx \quad$ and so

$$\lim_{b \to \infty} \int_0^{\sin b} \frac{1}{1 + u^2}\, du = \lim_{b \to \infty} \left[\tan^{-1} u \right]_0^{\sin b}$$

$$= \lim_{b \to \infty} \left(\tan^{-1}(\sin b) - 0 \right) \rightarrow \boxed{\text{Does not exist, Diverges}}$$

8. $$\int_0^{\frac{\pi}{2}} \frac{1}{1 - \cos x}\, dx$$

$$= \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \frac{1}{1 - \cos x}\, dx$$

$$= \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \frac{1}{1 - \cos x} \frac{(1 + \cos x)}{(1 + \cos x)}\, dx$$

$$= \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \frac{(1 + \cos x)}{1 - \cos^2 x}\, dx = \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \frac{(1 + \cos x)}{\sin^2 x}\, dx$$

$$= \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\, dx + \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x}\, dx$$

$$= \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \csc^2 x\, dx + \lim_{b \to 0^+} \int_b^{\frac{\pi}{2}} \cot x \csc x\, dx$$

$$= \lim_{b \to 0^+} \left[-\cot x \right]_b^{\frac{\pi}{2}} + \lim_{b \to 0^+} \left[-\csc x \right]_b^{\frac{\pi}{2}}$$

$$= -\cot\left(\frac{\pi}{2}\right) + \lim_{b \to 0^+} \cot b + -\csc\left(\frac{\pi}{2}\right) + \lim_{b \to 0^+} \csc b$$

$$= 0 + \infty - 1 + \infty \rightarrow \boxed{\infty,\ \text{Diverges}}$$