8.4 Partial Fractions

Partial Fraction Decomposition for $\dfrac{f(x)}{g(x)}$

1. If the degree of $g(x)$ is less than or equal to the degree of $f(x)$, use long division
2. Express $g(x)$ in its most compact factored form
3. For each factor $(ax+b)^n$, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

For each factor $(ax^2+bx+c)^n$, the partial fraction decomposition contains a sum of n partial fractions of the form

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Let's set one up $\rightarrow$ $\displaystyle\int \frac{3x^2+7x-11}{x^3(x^2+5)^3(x-1)^2(x^2+x+1)(x+3)^2}\,dx$

$$\int\left(\frac{Ax+B}{x^2+5} + \frac{Cx+D}{(x^2+5)^2} + \frac{Ex+F}{(x^2+5)^3} + \frac{G}{x-1} + \frac{H}{(x-1)^2} + \frac{Ix+J}{x^2+x+1} + \frac{K}{x+3} + \frac{L}{(x+3)^2} + \frac{M}{x} + \frac{N}{x^2} + \frac{P}{x^3}\right)dx$$

For problems 1 – 6, evaluate the integral.

1. $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$

$= \displaystyle\int \frac{4x^2+13x-9}{x(x+3)(x-1)}\,dx$

$= \displaystyle\int\left(\frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}\right)dx$

$4x^2+13x-9$

$= A(x+3)(x-1) + Bx(x-1) + Cx(x+3)$

$x=0 \rightarrow -9 = -3A \quad A = 3$

$x=1 \rightarrow 8 = 4C \quad C = 2$

$x=-3 \rightarrow -12 = 12B \quad B = -1$ so

$\displaystyle\int\left(\frac{3}{x} - \frac{1}{x+3} + \frac{2}{x-1}\right)dx$

$= 3\ln|x| - \ln|x+3| + 2\ln|x-1| + C$

2. $\displaystyle\int \frac{x^2-x-21}{2x^3-x^2+8x-4}\,dx$

$= \displaystyle\int \frac{x^2-x-21}{x^2(2x-1)+4(2x-1)}\,dx$

$= \displaystyle\int \frac{x^2-x-21}{(x^2+4)(2x-1)}\,dx$

$= \displaystyle\int\left(\frac{Ax+B}{x^2+4} + \frac{C}{2x-1}\right)dx$ so

$x^2-x-21 = (Ax+B)(2x-1) + C(x^2+4)$

$x = \frac{1}{2} \rightarrow \frac{-85}{4} = \frac{17}{4}C \rightarrow C = -5$

$1 = 2A + C \rightarrow A = 3$

$-21 = -B - 20 \rightarrow B = 1$

$\displaystyle\int\left(\frac{3x}{x^2+4} + \frac{1}{x^2+4} - \frac{5}{2x-1}\right)dx =$

$\frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{5}{2}\ln|2x-1| + C$

3. $\displaystyle\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}\,dx$

4. $\displaystyle\int \frac{-19x^2+50x-25}{x^2(3x-5)}\,dx$

$$= \int \left(\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2} \right) dx \quad \text{so}$$

$$5x^3 - 3x^2 + 7x - 3 = (Ax + B)(x^2 + 1) + (Cx + D)$$

$5 = A$ and $-3 = B$ and

$7 = A + C$ so $C = 2$ and $-3 = B + D$

so $D = 0$ and

$$= \int \left(\frac{5x}{x^2 + 1} - \frac{3}{x^2 + 1} + \frac{2x}{(x^2 + 1)^2} \right) dx$$

$$= \boxed{\frac{5}{2} \ln(x^2 + 1) - 3 \tan^{-1} x - \frac{1}{1 + x^2} + C}$$

5. $\displaystyle \int \frac{x^3 + 6x^2 + 3x + 16}{x^3 + 4x}\, dx$

$$\begin{array}{r|l} & 1 \\ \hline x^3 + 4x & x^3 + 6x^2 + 3x + 16 \\ & -(x^3 \qquad\quad + 4x) \\ \hline & 6x^2 - x + 16 \end{array} \quad \text{so}$$

$$= \int \left(1 + \frac{6x^2 - x + 16}{x^3 + 4x} \right) dx$$

$$= \int \left(1 + \frac{6x^2 - x + 16}{x(x^2 + 4)} \right) dx$$

$$= \int \left(1 + \frac{A}{x} + \frac{Bx + C}{x^2 + 4} \right) dx \quad \text{so}$$

$$6x^2 - x + 16 = A(x^2 + 4) + (Bx + C)x$$

$x = 0 \;\rightarrow\; 16 = 4A \;\rightarrow\; A = 4$

$6 = A + B \;\rightarrow\; B = 2$ and

$-1 = C$ so

$$= \int \left(1 + \frac{4}{x} + \frac{2x}{x^2 + 4} - \frac{1}{x^2 + 4} \right) dx$$

$$= \boxed{x + 4 \ln|x| + \ln(x^2 + 4) - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C}$$

$$= \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{3x - 5} \right) dx$$

$$-19x^2 + 50x - 25 = Ax(3x - 5) + B(3x - 5) + Cx^2$$

$x = 0 \;\rightarrow\; -25 = -5B \;\rightarrow\; B = 5$

$50 = -5A + 3B \;\rightarrow\; A = -7$

$-19 = 3A + C \;\rightarrow\; C = 2$

$$= \int \left(\frac{-7}{x} + \frac{5}{x^2} + \frac{2}{3x - 5} \right) dx$$

$$= \boxed{-7 \ln|x| - \frac{5}{x} + \frac{2}{3} \ln|3x - 5| + C}$$

6. $\displaystyle \int \frac{4x^3 - 3x^2 + 6x - 27}{x^4 + 9x^2}\, dx$

$$= \int \frac{4x^3 - 3x^2 + 6x - 27}{x^2(x^2 + 9)}\, dx$$

$$= \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9} \right) dx \quad \text{so}$$

$$4x^3 - 3x^2 + 6x - 27$$

$$= Ax(x^2 + 9) + B(x^2 + 9) + (Cx + D)x^2$$

$x = 0 \;\rightarrow\; -27 = 9B \;\rightarrow\; B = -3$

$-3 = B + D \;\rightarrow\; D = 0$ $\quad 6 = 9A \;\rightarrow\; A = \frac{2}{3}$

$4 = C + A \;\rightarrow\; C = \frac{10}{3}$

$$= \int \left(\frac{\frac{2}{3}}{x} - \frac{3}{x^2} + \frac{\frac{10}{3}x}{x^2 + 9} \right) dx$$

$$= \boxed{\frac{2}{3} \ln|x| + \frac{3}{x} + \frac{5}{3} \ln(x^2 + 9) + C}$$