Chapter 8 Test   No Calculators   February 10, 2014   Name

1. Evaluate $\lim\limits_{x \to 0^+} \csc\left(\frac{x}{2}\right) \tan^{-1}(3x)$

$$\lim_{x\to 0^+} \frac{\tan^{-1}(3x)}{\sin\left(\frac{x}{2}\right)} \Rightarrow \frac{0}{0} \quad \underset{\longrightarrow}{L'} \quad \lim_{x\to 0^+} \frac{\frac{3}{1+9x^2}}{\frac{1}{2}\cos\left(\frac{x}{2}\right)} \longrightarrow \frac{3}{\frac{1}{2}} = \boxed{6}$$

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2. Evaluate $\lim\limits_{x \to 0^+} \frac{\log_5(3x)}{\cot x}$

$$\lim_{x\to 0^+} \frac{\log_5(3x)}{\cot x} \Rightarrow \frac{-\infty}{\infty} \quad \underset{\longrightarrow}{L'} \quad \lim_{x\to 0^+} \frac{\frac{1(3)}{(\ln 5)(3x)}}{-\csc^2 x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x \ln 5} \Rightarrow \frac{0}{0}$$

$$\underset{\longrightarrow}{L'} \quad \lim_{x\to 0} \frac{-2\sin x \cos x}{\ln 5} \longrightarrow \boxed{0}$$

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3. Evaluate $\lim\limits_{x \to 2^+} (\ln(x-1))^{x-2}$

$\lim\limits_{x\to 2^+} (\ln(x-1))^{x-2} \Rightarrow 0^0$ We need a new limit: $\lim\limits_{x\to 2^+} (x-2)\ln(\ln(x-1))$

$$\lim_{x\to 2^+} \frac{\ln(\ln(x-1))}{(x-2)^{-1}} \Rightarrow \frac{-\infty}{\infty} \quad \underset{\longrightarrow}{L'} \quad \lim_{x\to 2^+} \frac{\frac{1}{\ln(x-1)} \cdot \frac{1}{x-1}}{-(x-2)^{-2}} = \lim_{x\to 2^+} \frac{-(x-2)^2}{(x-1)\ln(x-1)} \Rightarrow \frac{0}{0}$$

$$\underset{\longrightarrow}{L'} \quad \lim_{x\to 2^+} \frac{-2(x-2)}{\ln(x-1)+1} \longrightarrow 0 \qquad e^0 = \boxed{1}$$

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4. Evaluate $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} (2 + \cos(2x))^{\tan x}$

$\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} (2+\cos(2x))^{\tan x} \longrightarrow 1^\infty$ We need a new limit: $\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\ln(2+\cos 2x)}{\cot x}$

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\ln(2+\cos 2x)}{\cot x} \Rightarrow \frac{0}{0} \quad \underset{\longrightarrow}{L'} \quad \lim_{x\to\left(\frac{\pi}{2}\right)^-} \frac{\frac{1}{2+\cos 2x}(-2\sin 2x)}{-\csc^2 x} \longrightarrow \frac{\frac{-2(0)}{2+(-1)}}{-1} = \frac{0}{-1} = 0 \qquad e^0 = \boxed{1}$$

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5. Let $f(x) = \log_3 x^4$ and $g(x) = \ln\sqrt{5x}$ Which of the following is/are true? Show your work

I. $f = o(g)$ II. $f = O(g)$ III. $g = o(f)$ IV. $g = O(f)$

$$\lim_{x\to\infty} \frac{4\log_3 x}{\frac{1}{2}\ln(5x)} = \lim_{x\to\infty} \frac{4\frac{\ln x}{\ln 3}}{\frac{1}{2}(\ln 5 + \ln x)} = \lim_{x\to\infty} \frac{4\ln x}{\frac{1}{2}\ln 3(\ln 5 + \ln x)} = \frac{8}{\ln 3}$$

Since the two functions grow at the same rate, **II and IV** are correct

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6. Evaluate $\lim\limits_{x \to 0} (\cos x)^{\frac{1}{x^2}}$

$\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x^2}} \longrightarrow 1^\infty$ We need a new limit: $\lim\limits_{x\to 0} \frac{\ln(\cos x)}{x^2} \longrightarrow \frac{0}{0}$

$$\underset{\longrightarrow}{L'} \quad \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x} = \lim_{x\to 0} \frac{-\tan x}{2x} \longrightarrow \frac{0}{0} \quad \underset{\longrightarrow}{L'} \quad \lim_{x\to 0} \frac{-\sec^2 x}{2} \longrightarrow \frac{-1}{2} \qquad \boxed{e^{\frac{-1}{2}} = \frac{1}{\sqrt{e}}}$$

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7. Evaluate $\int_0^{\frac{2}{\pi}} \frac{\pi}{x^2} \cos\left(\frac{1}{x}\right) dx$

$\lim_{b \to 0^+} \int_0^{\frac{2}{\pi}} \frac{\pi}{x^2} \cos\left(\frac{1}{x}\right) dx$ Using u – substitution: $u = \frac{1}{x}$ $du = \frac{-1}{x^2} dx$ $-\pi du = \frac{\pi}{x^2} dx$

$-\pi \lim_{b \to 0^+} \int_{\frac{1}{b}}^{\frac{\pi}{2}} \cos(u)\, du = -\pi \lim_{b \to 0^+} \left[\sin u\right]_{\frac{1}{b}}^{\frac{\pi}{2}} = -\pi\left(1 - \lim_{b \to 0^+} \sin\left(\frac{1}{b}\right)\right) \longrightarrow$ **Oscillates, so DNE → diverges**

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8. Evaluate $\int_0^{-\infty} \cos(3x)\, e^x\, dx$

Integrating by parts, $u = \cos 3x$; $du = -3\sin(3x)\,dx$; $v = e^x$; $dv = e^x\,dx$ $\int \cos(3x)\, e^x\, dx = \cos(3x)\, e^x + 3\int \sin(3x)\, e^x\, dx$

Integrating by parts again to evaluate $\int \sin(3x)\, e^x\, dx$, $u = \sin 3x$; $du = 3\cos(3x)\,dx$, $v = e^x$, $dv = e^x\,dx$

$\int \sin(3x)\, e^x\, dx = \sin(3x)\, e^x - 3\int \cos(3x)\, e^x\, dx$ so $\int \cos(3x)\, e^x\, dx = \cos(3x)\, e^x + 3\left[\sin(3x)\, e^x - 3\int \cos(3x)\, e^x\, dx\right]$

Letting $A = \int \cos(3x)\, e^x\, dx$, $A = \cos(3x)\, e^x + 3\sin(3x)\, e^x - 9A$ so $10A = \cos(3x)\, e^x + 3\sin(3x)\, e^x$

$\lim_{b \to -\infty} \left[\frac{1}{10}\cos(3x)\, e^x + \frac{3}{10}\sin(3x)\, e^x\right]_0^b \Longrightarrow 0 - \left(\frac{1}{10} + 0\right) =$ **$\frac{-1}{10}$ → converges**

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9. Evaluate $\int_1^3 \frac{6x}{\sqrt[3]{x^2 - 9}} dx$

$\int_1^3 \frac{6x}{\sqrt[3]{x^2 - 9}} dx = \lim_{b \to 3^-} \int_1^b \frac{6x}{(x^2 - 9)^{\frac{1}{3}}} dx$ Using u – substitution, $u = x^2 - 9$, $du = 2x\,dx$, $3\,du = 6x\,dx$

$\lim_{b \to 3^-} \int_1^b \frac{6x}{(x^2 - 9)^{\frac{1}{3}}} dx = 3 \lim_{b \to 3^-} \int_{-8}^{b^2 - 9} u^{\frac{-1}{3}} du = 3\left(\frac{3}{2}\right) \lim_{b \to 3^-} \left[u^{\frac{2}{3}}\right]_{-8}^{b^2 - 9} = \frac{9}{2}\left[\lim_{b \to 3^-} (b^2 - 9)^{\frac{2}{3}} - 4\right] =$ **-18 → converges**

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10. Use the Direct Comparison Test or the Limit Comparison Test to determine if the following integral Converges or Diverges. Be sure to choose a related, simpler, p – value integral for comparison. $\int_0^3 \frac{2}{x^2 + 3x + \sqrt{x}} dx$

$\int_0^3 \frac{2}{x^2 + 3x + \sqrt{x}} dx \le \frac{2}{\sqrt{x}}$ $2\int_0^3 \frac{1}{x^{\frac{1}{2}}} dx = 4 \lim_{b \to 0^-} \left[x^{\frac{1}{2}}\right]_b^3 = 4\sqrt{3}$ → converges, so the original integral also **converges**

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11. Evaluate $\displaystyle\int_{-1}^{\infty} \frac{2x}{(1 + x^2)^{\frac{4}{3}}}\,dx$

Using u – sub, $u = 1 + x^2$,

$$du = 2x\,dx \qquad \int_{-1}^{\infty} \frac{2x}{(1 + x^2)^{\frac{4}{3}}}\,dx = \lim_{b\to\infty} \int_{2}^{1+b^2} u^{\frac{-4}{3}}\,du = \lim_{b\to\infty} \left[-3u^{\frac{-1}{3}}\right]_{2}^{1+b^2} = \lim_{b\to\infty}\left(\frac{-3}{\sqrt[3]{1 + b^2}} + \frac{3}{\sqrt[3]{2}}\right)$$

$$= \frac{3}{\sqrt[3]{2}} = \boxed{\frac{3\sqrt[3]{4}}{2} \rightarrow \text{converges}}$$

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12. Evaluate $\displaystyle\int \frac{x^2 + 3x + 1}{(x^2 + 1)(x^2 + 4)}\,dx$

$$\int \frac{x^2 + 3x + 1}{(x^2 + 1)(x^2 + 4)}\,dx = \int\left[\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}\right]dx \quad \text{so} \quad x^2 + 3x + 1 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1)$$

$1 = B + D$; $3 = 4A + C$; $1 = 4B + D$; $0 = A + C$ $\longrightarrow$ Solving the system of equations, $A = 1,\ B = 0,\ C = -1,\ D = 1$

$$\int\left[\frac{x}{x^2 + 1} - \frac{x}{x^2 + 4} + \frac{1}{x^2 + 4}\right]dx = \boxed{\frac{1}{2}\ln(x^2 + 1) - \frac{1}{2}\ln(x^2 + 4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$
