Quadratic Expressions

$$\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$$

For problems 1 – 6, evaluate the integral.

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1. $\displaystyle\int \frac{1}{9x^2 + 6x + 5}\,dx$

$$= \int \frac{1}{(9x^2 + 6x + 1) + 5 - 1}\,dx$$

$$= \int \frac{1}{(3x + 1)^2 + 2^2}\,dx$$

$3x + 1 = 2\tan\theta \qquad dx = \frac{1}{3}\,2\sec^2\theta\,d\theta$

$$\rightarrow \int \frac{\frac{2}{3}\sec^2\theta\,d\theta}{4\tan^2\theta + 4} = \frac{1}{6}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{6}\theta + C = \boxed{\frac{1}{6}\tan^{-1}\left(\frac{3x + 1}{2}\right) + C}$$

2. $\displaystyle\int \frac{1}{\sqrt{9 + 16x - 4x^2}}\,dx$

$$= \int \frac{1}{\sqrt{16 + 9 - (4x^2 - 16x + 16)}}\,dx$$

$$= \int \frac{1}{\sqrt{25 - (2x - 4)^2}}\,dx$$

$\rightarrow \quad u = 2x - 4 \qquad \frac{1}{2}\,du = dx$

$$\rightarrow \frac{1}{2}\int \frac{1}{\sqrt{5^2 - u^2}}\,du$$

$$= \boxed{\frac{1}{2}\sin^{-1}\left(\frac{2x - 4}{5}\right) + C}$$

3. $\int x\sqrt{3 - 2x - x^2}\, dx$

$$= \int x\sqrt{1 + 3 - (x^2 + 2x + 1)}\, dx$$

$$= \int x\sqrt{4 - (x+1)^2}\, dx$$

$x + 1 = 2\sin\theta \qquad dx = 2\cos\theta\, d\theta$

$$\rightarrow \int (2\sin\theta - 1)\sqrt{4 - 4\sin^2\theta}\; 2\cos\theta\, d\theta$$

$$= 8\int \sin\theta\cos^2\theta\, d\theta - 4\int \cos^2\theta\, d\theta$$

$$= \frac{-8}{3}\cos^3\theta - 2\theta - \sin 2\theta + C$$

$$= \frac{-8}{3}\cos^3\theta - 2\theta - 2\sin\theta\cos\theta + C$$

$$= \frac{-8}{3}\left(\frac{\sqrt{4-(x+1)^2}}{2}\right)^3 - 2\sin^{-1}\left(\frac{x+1}{2}\right) - 2\left(\frac{x+1}{2}\right)\left(\frac{\sqrt{4-(x+1)^2}}{2}\right) + C$$

$$= \frac{-1}{3}(3 - 2x - x^2)^{\frac{3}{2}} - 2\sin^{-1}\left(\frac{x+1}{2}\right) - (x+1)\left(\frac{\sqrt{3 - 2x - x^2}}{2}\right) + C$$

4. $\int \frac{2x - 1}{x^2 - 6x + 13}\, dx$

$$= \int \frac{2x - 1}{(x^2 - 6x + 9) + 13 - 9}\, dx$$

$$= \int \frac{2x - 1}{(x-3)^2 + 4}\, dx$$

$u = x - 3 \qquad du = dx \qquad x = u + 3$

$$\rightarrow \int \frac{2u + 5}{u^2 + 4}\, du$$

$$= \int \frac{2u}{u^2 + 4}\, du + \int \frac{5}{u^2 + 4}\, du$$

$$= \ln(u^2 + 4) + \frac{5}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \ln(x^2 - 6x + 13) + \frac{5}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C$$

5. $\int \frac{1}{\sqrt{2x - x^2}}\, dx$

$$= \int \frac{1}{\sqrt{) + 1 - (x^2 - 2x + 1)}}\, dx$$

$$= \int \frac{1}{\sqrt{1 - (x-1)^2}}\, dx$$

$u = x - 1 \qquad du = dx$

$$\rightarrow \int \frac{1}{\sqrt{1 - u^2}}\, du = \sin^{-1}(x - 1) + C$$

6. $\int \frac{x}{\sqrt{5 + 12x - 9x^2}}\, dx$

$$= \int \frac{x}{\sqrt{5 + 4 - (9x^2 - 12x + 4)}}\, dx$$

$$= \int \frac{x}{\sqrt{9 - (3x-2)^2}}\, dx$$

$3x - 2 = 3\sin z \qquad 3\,dx = 3\cos z\, dz$

$dx = \cos z\, dz \qquad x = \sin z + \frac{2}{3}$

$$\rightarrow \int \frac{\sin z + \frac{2}{3}}{\sqrt{9 - 9\sin^2 z}}\cos z\, dz$$

$$= \int \frac{\sin z + \frac{2}{3}}{3\cos z}\cos z\, dz = \frac{1}{3}\int\left(\sin z + \frac{2}{3}\right) dz$$

$$= \frac{-1}{3}\cos z + \frac{2}{9}z + C$$

$$= \frac{-1}{3}\,\frac{\sqrt{5 + 12x - 9x^2}}{3} + \frac{2}{9}\sin^{-1}\left(\frac{3x-2}{3}\right) + C$$

$$= \frac{-\sqrt{5 + 12x - 9x^2}}{9} + \frac{2}{9}\sin^{-1}\left(\frac{3x-2}{3}\right) + C$$