Trigonometric Product Integrals

For problems 1 – 4, evaluate the integral.

1. $\int \tan^3 x \sec^5 x \, dx$

$= \int \tan^2 x \sec^4 x \, (\sec x \tan x \, dx)$

$= \int (\sec^2 x - 1) \sec^4 x \, (\sec x \tan x \, dx)$

$= \int (\sec^6 x - \sec^4 x) \, (\sec x \tan x \, dx)$

$u = \sec x \qquad du = \sec x \tan x \, dx$

$\rightarrow \int (u^6 - u^4) \, du = \frac{1}{7} u^7 - \frac{1}{5} u^5 + C$

$= \boxed{\frac{1}{7} \sec^7 x - \frac{1}{5} \sec^5 x + C}$

2. $\int \sin^5 x \cos^3 x \, dx$

$= \int \sin^5 x \cos^2 x \, (\cos x \, dx)$

$= \int \sin^5 x \, (1 - \sin^2 x) \, (\cos x \, dx)$

$= \int (\sin^5 x - \sin^7 x) \, (\cos x \, dx)$

$u = \sin x \qquad du = \cos x \, dx$

$\rightarrow \int (u^5 - u^7) \, du$

$= \boxed{\frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x + C}$

3. $\int \csc^6 (2x) \, dx$

$w = 2x \qquad \frac{1}{2} dw = dx \qquad \rightarrow \frac{1}{2} \int \csc^6 w \, dw$

$= \frac{1}{2} \int \csc^4 w \, (\csc^2 w \, dw)$

$= \frac{1}{2} \int (\cot^2 w + 1)^2 \, (\csc^2 w \, dw)$

$= \frac{1}{2} \int (\cot^4 w + 2 \cot^2 w + 1) (\csc^2 w \, dw)$

$u = \cot w \qquad -du = \csc^2 w \, dw$

$= \frac{-1}{2} \int (u^4 + 2u^2 + 1) \, du$

$= \boxed{\frac{-1}{2} \left(\frac{1}{5} \cot^5 (2x) + \frac{2}{3} \cot^3 (2x) + \cot (2x) \right) + C}$

4. $\int \sec^3 x \, dx$

$u = \sec x \qquad du = \sec x \tan x \, dx$

$v = \tan x \qquad dv = \sec^2 x \, dx$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x \, (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \ln \left| \sec x + \tan x \right| \qquad$ so

$Q = \int \sec^3 x \, dx = \sec x \tan x - Q + \ln \left| \sec x + \tan x \right|$

$\int \sec^3 x \, dx = \boxed{\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \left| \sec x + \tan x \right| + C}$