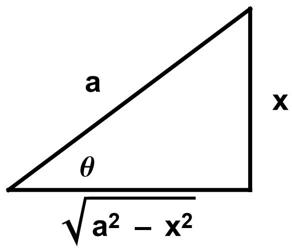
Trigonometric Substitution

$a^2 - x^2$

$x = a\sin\theta$

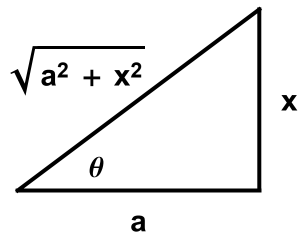
$\sin\theta = \dfrac{x}{a}$



$a^2 + x^2$

$x = a\tan\theta$

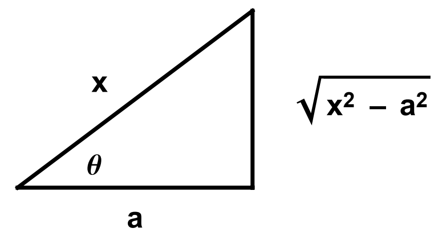
$\tan\theta = \dfrac{x}{a}$



$x^2 - a^2$

$x = a\sec\theta$

$\sec\theta = \dfrac{x}{a}$



For problems 1 – 6, evaluate the integral.

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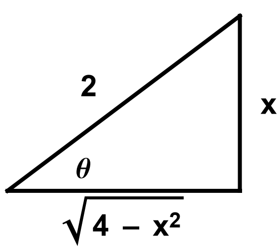
1. $\displaystyle\int \frac{x^3}{\sqrt{4 - x^2}}\,dx$



$x = 2\sin\theta \qquad dx = 2\cos\theta\, d\theta$

$$= \int \frac{8\sin^3\theta}{\sqrt{4 - 4\sin^2\theta}}\,(2\cos\theta)\,d\theta$$

$$= \int \frac{8\sin^3\theta}{(2\cos\theta)}\,(2\cos\theta)\,d\theta$$

$$= \int 8\sin^3\theta\,d\theta$$

$$= 8\int (1 - \cos^2\theta)(\sin\theta\,d\theta)$$

$u = \cos\theta \qquad -du = \sin\theta\,d\theta$

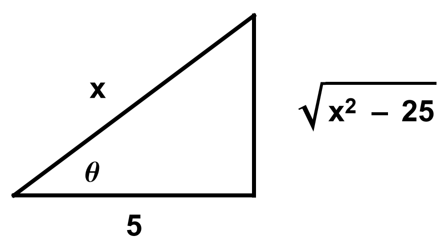
$$= -8\int (1 - u^2)\,du \quad = -8\left(u - \frac{1}{3}u^3\right) + C$$

$$= -8\cos\theta + \frac{8}{3}\cos^3\theta + C$$

$$= -8\left(\frac{\sqrt{4 - x^2}}{2}\right) + \frac{8}{3}\left(\frac{\sqrt{4 - x^2}}{2}\right)^3 + C$$

$$= \boxed{-4\sqrt{4 - x^2} + \frac{1}{3}(4 - x^2)^{\frac{3}{2}} + C}$$

2. $\displaystyle\int \frac{\sqrt{x^2 - 25}}{x}\,dx$



$x = 5\sec\theta \qquad dx = 5\sec\theta\tan\theta\,d\theta$

so $\displaystyle\int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}\,d\theta$

$$= \int \frac{5\tan\theta}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta \quad = \int 5\tan^2\theta\,d\theta$$

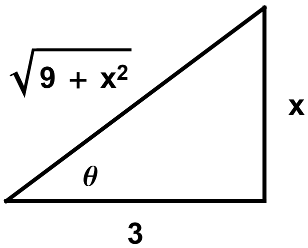
$$= 5\int (\sec^2\theta - 1)\,d\theta = 5\tan\theta - 5\theta + C$$

$$= 5\left(\frac{\sqrt{x^2 + 25}}{5}\right) - 5\sec^{-1}\left(\frac{x}{5}\right) + C$$

$$= \boxed{\sqrt{x^2 - 25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C}$$

3. $\int \frac{x^2}{\sqrt{9+x^2}}\,dx$

4. $\int \frac{1}{(4x^2-1)^{\frac{3}{2}}}\,dx$



$x = 3\tan\theta \quad dx = 3\sec^2\theta\,d\theta$

$$\to \int \frac{9\tan^2\theta}{\sqrt{9+9\tan^2\theta}}\,3\sec^2\theta\,d\theta$$

$$= \int \frac{9\tan^2\theta}{3\sec\theta}\,3\sec^2\theta\,d\theta \quad = \quad 9\int \tan^2\theta\sec\theta\,d\theta$$

$u = \tan\theta \qquad du = \sec^2\theta\,d\theta$

$v = \sec\theta \qquad dv = \sec\theta\tan\theta\,d\theta$

$$= \quad 9\left(\sec\theta\tan\theta - \int \sec^3\theta\,d\theta\right)$$

$$= \quad 9\left(\sec\theta\tan\theta - \int (\tan^2\theta + 1)\sec\theta \;\; d\theta\right)$$

$$= \quad 9\left(\sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta - \int \sec\theta\,d\theta\right)$$

$$= \quad 9\left(\sec\theta\tan\theta - \int \tan^2\theta\sec\theta\,d\theta - \ln\left|\sec\theta + \tan\theta\right|\right)$$

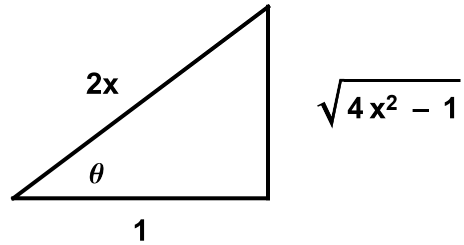
$Q = 9\sec\theta\tan\theta - Q - 9\ln|\sec\theta + \tan\theta|$ or

$2Q = 9\sec\theta\tan\theta - 9\ln|\sec\theta + \tan\theta|$ so

$$Q = \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{9}{2}\left(\frac{\sqrt{x^2+9}}{3}\right)\left(\frac{x}{3}\right) - \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

$$= \boxed{\frac{x\sqrt{x^2+9}}{2} - \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right| + C}$$



$2x = \sec\theta \qquad dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$

$$\int \frac{\frac{1}{2}\sec\theta\tan\theta\,d\theta}{(\sec^2\theta - 1)^{\frac{3}{2}}} = \frac{1}{2}\int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta}$$

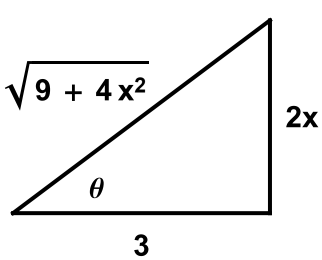
$$= \frac{1}{2}\int \frac{\cos^2\theta}{\cos\theta\,\sin^2\theta}\,d\theta = \frac{1}{2}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{2}\int \csc\theta\cot\theta\,d\theta = \frac{-1}{2}\csc\theta + C$$

$$= \boxed{\frac{-1}{2}\left(\frac{2x}{\sqrt{4x^2-1}}\right) + C = \frac{-x}{\sqrt{4x^2-1}} + C}$$

5. $\int \frac{1}{(4x^2+9)^2}\,dx$

6. $\int \frac{3x}{\sqrt{49-9x^2}}\,dx$



No need to do a trig substitution here,

$2x = 3\tan\theta \qquad dx = \frac{3}{2}\sec^2\theta \; d\theta$

$$\rightarrow \int \frac{\frac{3}{2}\sec^2\theta \; d\theta}{(9\tan^2\theta + 9)^2}$$

$$= \int \frac{\frac{3}{2}\sec^2\theta \; d\theta}{81\sec^4\theta} = \int \frac{1}{54}\cos^2\theta \, d\theta$$

$$= \frac{1}{108}\int (1 + \cos 2\theta)\, d\theta = \frac{1}{108}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{108}(\theta + \sin\theta\cos\theta) + C$$

$$\frac{1}{108}\left(\tan^{-1}\left(\frac{2x}{3}\right) + \frac{6x}{4x^2 + 9}\right) + C$$

$$= \boxed{\frac{1}{108}\tan^{-1}\left(\frac{2x}{3}\right) + \frac{x}{18(4x^2 + 9)} + C}$$

$u = 49 - 9x^2 \qquad du = -18x\,dx \quad \rightarrow \quad \frac{-1}{6}du = 3x\,dx$

$$\text{so} \quad \frac{-1}{6}\int u^{\frac{-1}{2}}\, du = \frac{-1}{6}\left(2\sqrt{u}\right) + C$$

$$\rightarrow \boxed{\frac{-1}{3}\sqrt{49 - 9x^2} + C}$$
