9.1 Differentiation and Integration of Power Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{where} \quad -1 < x < 1$$

Series Differentiation

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Then $f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1}$

Series Integration

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Then $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

For problems 1 and 2, express as a ratio of two integers.

1. .863636363...

$= 0.8 + (0.063 + 0.00063 + 0.0000063 + \ldots)$

$r = \frac{1}{100} \quad a = 0.063 \quad$ so

$S = \frac{\frac{63}{1000}}{1 - \frac{1}{100}} = \frac{63}{990} = \frac{7}{110}$ and

$\frac{4}{5} + \frac{7}{110} = \frac{88+7}{110} = \frac{95}{110} = \boxed{\frac{19}{22}}$

2. .3251251251...

$= 0.3 + (0.0251 + 0.0000251 + 0.0000000251 + \ldots)$

$r = \frac{1}{1000} \quad a = \frac{251}{10000}$

$S = \frac{\frac{251}{10000}}{1 - \frac{1}{1000}} = \frac{251}{9990}$ and

$\frac{3}{10} + \frac{251}{9990} = \frac{2997+251}{9990} = \frac{3248}{9990} = \boxed{\frac{1624}{4995}}$

For problems 3 – 10, find a power series to represent the given function, and the interval of convergence.

3. $\frac{2}{1+5x}$

$= 2\left(\frac{1}{1-(-5x)}\right) = 2\sum_{n=0}^{\infty} (-5x)^n$

$= \boxed{2\sum_{n=0}^{\infty} (-1)^n 5^n x^n} \qquad |-5x| < 1$

so $\boxed{-\frac{1}{5} < x < \frac{1}{5}}$

4. $\frac{x^2}{1+(x+3)}$

$= x^2\left(\frac{1}{1-(-(x+3))}\right)$

$= \boxed{\sum_{n=0}^{\infty} x^2 (-1)^n (x+3)^n} \qquad |x+3| < 1$

so $-1 < x+3 < 1$ or $\boxed{-4 < x < -2}$

5. $\ln(1+x)$

$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-1)^n x^n$ so

$\ln(1+x) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$ and

$\boxed{-1 < x \le 1}$

6. $\frac{3}{1+x^2}$

$= 3\left(\frac{1}{1-(-x^2)}\right) = \boxed{3\sum_{n=0}^{\infty} (-1)^n x^{2n}}$

and $\boxed{-1 < x < 1}$

7. $\tan^{-1} x$

Now, $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

so $$\tan^{-1} x = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$ and $\boxed{-1 \le x \le 1}$

8. $\frac{1}{4-3x}$

$$= \frac{1}{4}\left(\frac{1}{1-\frac{3}{4}x}\right) = \boxed{\frac{1}{4}\sum_{n=0}^{\infty} \frac{3^n x^n}{4^n}}$$

and $\left|\frac{3}{4}x\right| < 1$ so $\boxed{\frac{-4}{3} < x < \frac{4}{3}}$

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9. $\ln(x)$

Now, $$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

so $$\ln(x) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}}$$

and $|x-1| < 1$ or $\boxed{0 < x \le 2}$

10. $\frac{1}{(1-x)^2}$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$ and $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

so $$\frac{1}{(1-x)^2} = \boxed{\sum_{n=1}^{\infty} n x^{n-1}}$$ and $\boxed{-1 < x < 1}$
