9.1 Power Series

Infinite Series

$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ or $\sum_{k=1}^{\infty} a_k$

Geometric Series

$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots$ or $\sum_{n=1}^{\infty} ar^{n-1}$

I. Converges for $|r| < 1$, with a sum of $\frac{a}{1-r}$

II. Diverges if $|r| \geq 1$

Power Series I

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$ is a power series centered at $x = 0$

Power Series II

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots$

is a power series centered at $x = a$

For problems 1 – 3, rewrite using summation notation.

1. $\frac{2}{3} - \frac{2}{6} + \frac{2}{9} - \frac{2}{12} + \frac{2}{15} - \ldots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{3k}$

2. $71 + 0.71 + 0.0071 + 0.000071 + \ldots = \sum_{n=1}^{\infty} 71\left(\frac{1}{100}\right)^{n-1}$

3. $\frac{\pi}{2} + \frac{\pi^2}{4} + \frac{\pi^3}{8} + \frac{\pi^4}{16} + \ldots = \sum_{k=1}^{\infty} \left(\frac{\pi}{2}\right)^k$

For problems 4 – 7, determine whether the series converges or diverges. If it converges, give its sum.

4. $1 - \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 - \ldots$ $r = \frac{-3}{4}$ $a = 1$ $S = \frac{1}{1 - \left(\frac{-3}{4}\right)} = \frac{4}{7}$

5. $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{2} n\right) = 1 + 0 + (-1) + 0 + 1 + \ldots \rightarrow$ Diverges

6. $\sum_{k=0}^{\infty} \left(\frac{6}{5}\right)\left(\frac{1}{3}\right)^k$ $\quad r = \frac{1}{3} \quad a = \frac{6}{5} \quad S = \frac{\frac{6}{5}}{1 - \frac{1}{3}} = \frac{6}{5}\left(\frac{3}{2}\right) = \boxed{\frac{9}{5}}$

7. $0.4 - 0.04 + 0.004 - 0.0004 + \ldots$ $\quad r = \frac{-1}{10} \quad a = 0.4 \quad S = \frac{0.4}{1 - \left(\frac{-1}{10}\right)} = \boxed{\frac{4}{11}}$

For problems 8 – 10, find the interval of convergence and the function of x represented by the series.

8. $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{3^n}$ $\quad r = \frac{x+2}{-3} \quad \left|\frac{x+2}{-3}\right| < 1 \quad -1 < \frac{x+2}{-3} < 1$

$3 > x + 2 > -3 \qquad -3 < x + 2 < 3 \qquad \boxed{-5 < x < 1}$

$S = \frac{\frac{x+2}{-3}}{1 + \frac{x+2}{3}} = \frac{x+2}{-3}\left(\frac{3}{x+5}\right) = \boxed{\frac{-x-2}{x+5}}$

9. $\sum_{n=0}^{\infty} 4(2x+1)^n$ $\quad r = 2x + 1 \quad |2x+1| < 1 \quad -1 < 2x + 1 < 1 \quad \rightarrow \boxed{-1 < x < 0}$

$S = \frac{4}{1 - (2x+1)} = \boxed{\frac{-2}{x}}$

10. $\sum_{n=0}^{\infty} \frac{-(x-4)^n}{5^n}$ $\quad r = \frac{x-4}{5} \quad \left|\frac{x-4}{5}\right| < 1 \quad -5 < x - 4 < 5 \quad \boxed{-1 < x < 9}$

$S = \frac{-1}{1 - \left(\frac{x-4}{5}\right)} = -1\left(\frac{5}{9-x}\right) = \boxed{\frac{5}{x-9}}$

11. If $\sum_{n=1}^{\infty} a r^{n-1} = 7$ and $a = 3$, find r $\quad S = \frac{a}{1-r} \rightarrow 7 = \frac{3}{1-r} \rightarrow 1 - r = \frac{3}{7}$ so $\boxed{r = \frac{4}{7}}$

12. Find a power series for $\frac{1}{1-x}$

$$\begin{array}{r l}
 & 1 + x + x^2 + x^3 + \ldots \\
\hline
1 - x \,) & 1 \\
 & -(1 - x) \\
 & x \\
 & -(x - x^2) \\
 & x^2 \\
 & -(x^2 - x^3) \\
 & x^3
\end{array}$$

and so on . . . so

$\frac{1}{1-x} = \boxed{\sum_{n=0}^{\infty} x^n}$ for $\boxed{-1 < x < 1}$