9.2 Taylor Series

Maclaurin Series

Let $f(x)$ be a function that has derivatives of all orders on an open interval containing 0. Then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

and the partial sum $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ is the Maclaurin polynomial of order n.

Taylor Series

Let $f(x)$ be a function that has derivatives of all orders on an open interval containing a. Then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$

and the partial sum $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$ is the Taylor polynomial of order n for f at $x = a$.

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ for $-1 < x < 1$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $-\infty < x < \infty$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for $-\infty < x < \infty$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $-\infty < x < \infty$

$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ for $-1 < x \le 1$

$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $-1 \le x \le 1$

Assuming that $f(x)$ has derivatives of all orders, and that $f(x) = \sum_{n=0}^{\infty} a_n x^n$, derive the Maclaurin Series

(we're trying to find a formula to replace the a_n)

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$ so $f(0) = a_0$

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots$ so $f'(0) = a_1$

$f''(x) = 2a_2 + 3(2)a_3 x + 4(3)a_4 x^2 + 5(4)a_5 x^3 + \ldots$ so $f''(0) = 2!\,a_2$

$f'''(x) = 3!\,a_3 + 4(3)(2)a_4 x + 5(4)(3)a_5 x^2 + \ldots$ so $f'''(0) = 3!\,a_3$

$f^{(4)}(x) = 4!\,a_4 + 5!\,a_5 x + \ldots$ so $f^{(4)}(0) = 4!\,a_4$

$f^{(5)}(x) = 5!\,a_5 + \ldots$ so $f^{(5)}(0) = 5!\,a_5$

following the pattern, $\boxed{f^{(n)}(0) = n!\,a_n}$ or $\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$ so if

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Now, let's derive the Maclaurin Series for $f(x) = \cos x$

$f(x) = \cos x$ so $f(0) = 1$ and $\dfrac{f(0)}{0!} = 1$

$f'(x) = -\sin x$ so $f'(0) = 0$ and $\dfrac{f'(0)}{1!} = 0$

$f''(x) = -\cos x$ so $f''(0) = -1$ and $\dfrac{f''(0)}{2!} = \dfrac{-1}{2!}$

$f^3(x) = \sin x$ so $f^3(0) = 0$ and $\dfrac{f^3(0)}{3!} = 0$

$f^4(x) = \cos x$ so $f^4(0) = 1$ and $\dfrac{f^4(0)}{4!} = \dfrac{1}{4!}$

$f^5(x) = -\sin x$ so $f^5(0) = 0$ and $\dfrac{f^5(0)}{5!} = 0$

$f^6(x) = -\cos x$ so $f^6(0) = -1$ and $\dfrac{f^6(0)}{6!} = \dfrac{-1}{6!}$

$f^7(x) = \sin x$ so $f^7(0) = 0$ and $\dfrac{f^7(0)}{7!} = 0$

$f^8(x) = \cos x$ so $f^8(0) = 1$ and $\dfrac{f^8(0)}{8!} = \dfrac{1}{8!}$

so $$P_8(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8$$

And, the series would be $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

For problems 1 – 5, use the table of Maclaurin Series given on the other side of this page. Find the Maclaurin Series for the given function and determine the interval of convergence.

1. $\cos 3x$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n} x^{2n}}{(2n)!}$$

2. $\tan^{-1}\left(\dfrac{x}{4}\right)$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{x}{4}\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{4^{2n+1}(2n+1)}$$

3. $x^2 e^{2x}$

$$= \sum_{n=0}^{\infty} \frac{x^2 (2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+2}}{n!}$$

4. $\ln\left(1 - \dfrac{x^2}{2}\right)$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{-x^2}{2}\right)^n}{n} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n\,2^n}$$

5. $$\frac{2x^3}{1-5x} = 2x^3\left(\frac{1}{1-5x}\right) = 2\sum_{n=0}^{\infty} x^3 (5x)^n = 2\sum_{n=0}^{\infty} 5^n x^{n+3}$$