9.3 Taylor's Theorem

Taylor's Theorem with Remainder

If f has derivatives of all orders in an open interval containing a, then for each positive integer n and for each x in the open interval,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x OR

$$f(x) = P_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$ for some c between a and x

For problems 1 and 2, find the Taylor polynomial of order 4 for the function at $x = 0$, and use it to approximate the value of the function at $x = 0.1$

1. $\dfrac{-1}{(1+x)^2}$

2. $2\sin\left(\dfrac{\pi x}{3}\right)$

For problems 3 – 6, find the Maclaurin series for the function.

3. $\ln(1 - x^3)$

4. $\tan^{-1}(x^2) - x^2 + \dfrac{x^6}{3}$

5. 3^x

6. $\dfrac{x}{1 - 3x}$

7. If e^{2x} is approximated with $1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!}$, and $|x| < \frac{1}{2}$, what can be said about the error?

8. If $\cos\left(\frac{x}{3}\right)$ is approximated with $1 - \frac{x^2}{3^2(2!)} + \frac{x^4}{3^4(4!)}$, what range of x – values can be plugged in so that the error is no larger than 0.0001? Will this approximation be too large or too small?

9. If $\ln(1 + x)$ is approximated with $x - \frac{x^2}{2} + \frac{x^3}{3}$, and $|x| < \frac{1}{4}$, what can be said about the error? Will this approximation be too large or too small?

10. If $\sqrt[3]{1 + x}$ is approximated with $1 + \frac{x}{3}$, and $x = \frac{1}{2}$, what can be said about the error?