9.5 Testing For Convergence

The Integral Test

Let $\sum a_n$ be a series with positive terms, and suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \ge N$ (where N is some positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\, dx$ either both converge or both diverge.

The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{n} + \ldots$$

Diverges!

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p – series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ (a) Diverges for $p \le 1$ (b) Converges for $p > 1$

The Direct Comparison Test

Let $\sum a_n$ be a series with no negative terms

(a) $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \le c_n$ for all $n > N$, for some integer N

(b) $\sum a_n$ diverges if there is a divergent series $\sum d_n$ of nonnegative terms with $a_n \ge d_n$ for all $n > N$, for some integer N

The Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with positive terms

(a) if $\lim_{n \to \infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge

(b) if $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges also

(c) if $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges also

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For problems 1 – 4, determine whether the series converges or diverges.

1. $\sum_{n=0}^{\infty} \frac{2^n}{1 + 2^{2n}}$

2. $\sum_{n=1}^{\infty} \frac{4n^4 - 2n^3}{(n+1)(2n^2 - n)(3n + n^2)}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2} + 2}{n^2 + 1}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

The Alternating Series Test

The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + . . .$ **converges if all three of the following conditions are satisfied**

(a) each u_n **is positive** **(b)** $u_n \geq u_{n+1}$ **for all** $n \geq N$, **for some integer** N **(c)** $\displaystyle\lim_{n \to \infty} u_n = 0$

For problems 5 – 10, determine whether the series converges absolutely, converges conditionally, or diverges. Justify your answer.

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{\ln n}}$

6. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n} \frac{n!}{10^n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^{\frac{4}{3}}}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n\,3^n}{5^n}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(\ln 2)^n}$